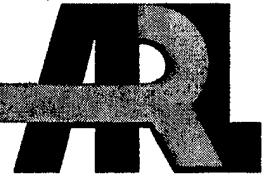ARMY RESEARCH LABORATORY

A Continuum Framework for Finite Viscoplasticity and Classes of Flow Rules for Finite Viscoplasticity

by Mike Scheidler and T. W. Wright

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14. ABSTRACT

A continuum framework for finite viscoplasticity is developed based on Lee's multiplicative decomposition with internal variables. Noteworthy features include a thermodynamically consistent treatment of the storage of cold work and plastic volume change and a careful examination of the restrictions imposed by the entropy inequality and the property of instantaneous thermoelastic response.

Classes of flow rules for finite viscoplasticity are defined by assuming that certain measures for plastic strain rate and plastic spin depend on the state variables but not on the plastic deformation. It is shown that three of these classes are mutually exclusive for finite elastic strains. For small elastic shear strains, two of the three classes are approximately equivalent. A number of exact and approximate kinematic relations between the various measures for plastic strain rate and plastic spin are derived. Some inconsistent flow rules encountered in the literature are also discussed. Throughout the paper, arbitrarily anisotropic materials are considered, and some of the simplifications resulting from the assumption of isotropy are noted.

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Preface

This report contains reprints of two journal papers. The first paper, "A Continuum Framework for Finite Viscoplasticity,"[1] provides a continuum thermodynamic framework for finite deformation viscoplasticity based on the multiplicative decomposition of the deformation gradient with internal state variables. The second paper, "Classes of Flow Rules for Finite Viscoplasticity,"[2] compares several classes of flow rules within the framework of the first paper.

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[2] Reprinted from the *International Journal of Plasticity*, vol. 19, Mike Scheidler and T. W. Wright, "Classes of Flow Rules for Finite Viscoplasticity," pp. 1119–1165, 2003, with permission from Elsevier.

A continuum framework for finite viscoplasticity

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Abstract

A continuum framework for finite viscoplasticity is developed based on Lee's multiplicative decomposition with internal variables. Noteworthy features include a thermodynamically consistent treatment of the storage of cold work and plastic volume change and a careful examination of the restrictions imposed by the entropy inequality and the property of instantaneous thermoelastic response. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

More than 30 years have passed since E.H. Lee (1969) introduced the idea of decomposing the deformation gradient into multiplicative elastic and plastic parts for truly finite deformations. The theory has developed on many fronts in that time and has been applied to many practical problems and calculations. Nevertheless, it seems fair to say that there is not yet a version of the theory that commands the same widespread acceptance routinely given to finite elasticity. It has seemed to us that the reason for this lack of universal agreement lies in the very foundations of the theory and therefore that a reexamination of those foundations would probably be worthwhile and clarifying for many issues. It is the purpose of this paper to begin that reexamination. Throughout the paper arbitrarily anisotropic materials are considered.

The discussion opens in Section 2 with a statement of what we conceive to be the primary physical basis for a theory of plastic deformation in crystalline bodies. The

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central observation is that even in the presence of very large numbers of dislocations and other defects, the average spacing of dislocations is much larger than lattice dimensions. Therefore, a plastically deformed body should remain essentially elastic in the way that stresses are generated from lattice extension and distortion and in the way that forces are transmitted to adjacent material. Defects further distort the material and give rise to locally self-equilibrated, but still elastic stresses. The total elastic energy must then include contributions from these equilibrated fluctuations as well as from the mean extension and distortion of the lattice. Rather than using the details or statistics of dislocation density, we choose to represent the contribution from fluctuations by internal variables that are designed to capture the phenomenology of plastic deformation.

The standard multiplicative decomposition of the deformation gradient is assumed, and some of the fundamental kinematic issues associated with the decomposition are raised in Section 3. The intermediate configuration is regarded as a reference configuration for finite elastic deformation, and the second Piola–Kirchhoff stress tensor is defined with respect to it. Plastic deformation is further divided into volumetric and isochoric parts to take explicit account of possible plastic volume changes. The effect of superposed rigid motions is examined next, and some of the principal transformation rules are established. In particular, the plastic part of the deformation gradient is regarded as invariant under rigid body motions. This last supposition is consistent with the notion that all the crystalline defects, which are the origin of plastic deformation, are invariant under rigid motions. In effect, the added rigid motion is regarded as part of the elastic deformation only. In one useful approximate case, the elastic shear strains are small, but elastic rotation and volume change may be large to allow for large rigid motion and high pressure.

Thermodynamic restrictions on material response are discussed in Section 4. First the balance laws for momentum and energy are introduced, and the total stress power is expressed in both the current and initial configurations. Then the reduced entropy inequality, based on a form often used in continuum mechanics, is stated in terms of the Helmholz free energy per unit mass. Next the stress power is partitioned into elastic and plastic parts, both of which may also be expressed in the current and intermediate configurations. This partitioning is essential for the reductions to follow.

At this point, two crucial assumptions are made regarding the functional dependencies of certain constitutive quantities and the evolution of key variables. First, the energies and Cauchy stress are assumed to be functions of the elastic deformation as measured from the intermediate configuration, the temperature, and the internal variables but not in any way dependent on the plastic part of the deformation. This assumption follows from the idea that stress is the consequence of an extended and distorted crystal lattice but not of dislocations (which are regarded as a consequence of lattice instability) or other manifestations of plasticity per se. Second, the evolution of variables is assumed to exhibit what may be called instantaneous thermoelastic response, and a physical argument is given to motivate the assumption. A somewhat technical definition is required, but in essence it states that acceleration waves are essentially elastic. These two assumptions are sufficient to

ensure that the internal energy and Helmholz free energy are potential functions for the second Piola–Kirchhoff stress and either temperature or entropy. The significance here is that the potential relationships are proved to exist even when the current stress lies outside the yield surface so that viscoplastic flow must be occurring. The assumption of instantaneous thermoelastic response is the key to the argument since that is what allows the jumps in the rates of elastic strain and temperature to be arbitrary and independent, thus requiring their multiplicative factors in the entropy inequality to vanish and in turn establishing the existence of the potential relationships.

The derivatives of the potentials with respect to the internal variables define conjugate internal forces that do work against changes in internal variables. A residual inequality immediately follows that states that the rate of working against the internal variables, and hence the rate of storage of cold work, can be no greater than the rate of plastic work. We also note that the usual inequality for heat conduction may be extended to hold during plastic as well as purely elastic deformation, although that does not follow as a consequence of the other assumptions, and we do not make further use of that extension. Another consequence of the assumption of instantaneous thermoelastic response is that density in the intermediate configuration can be at most a function of the internal variables, but cannot depend on either the elastic strain or the temperature.

The plastic stress power may be decomposed into contributions due to dislocation slip and plastic volume change, since plastic incompressibility has not been imposed a priori. Furthermore, if the intermediate density is chosen to be an internal variable itself, it is found that an effective pressure does work against the changing intermediate density. This effective pressure essentially consists of the difference between the usual external pressure and the internal force conjugate to the intermediate density. The existence of an effective pressure may have some bearing on the curious fact that metals that display a stress differential effect (different yield stresses in tension and compression) do not show the corresponding volumetric change that might have been expected from the normality rule.

The Gibbs function may be used as a potential for elastic strain and entropy, provided that the second Piola–Kirchhoff stress is invertible in elastic strain. Now stress and temperature are regarded as the primary variables, which is useful since stress rather than strain is often used as the primary mechanical variable in theories of plasticity. Section 4 closes with a brief discussion of conjugate stress and strain tensors. Potential relations hold for any elastic strain tensor and its conjugate stress tensor, and the concept of conjugate plastic strain rate is introduced.

The yield function and equations that control the evolution of the internal variables and plastic flow are introduced in Section 5. Starting from a general form with the evolutionary rates dependent on the plastic part of the deformation as well as the mechanical, thermal, and internal variables, various restrictions are deduced or imposed. Besides the usual invariance under rigid rotations, it is argued that the evolutionary rates of the internal variables should not depend on the isochoric part of the plastic deformation, although they may depend on the volumetric part since the intermediate density may itself be an internal variable. The basic notion for this

exclusion is that plastic slip is not an adequate measure of any aspect of the state of the material even though it is the fundamental process that generates changes in the internal variables. It is argued further that determination of the plastic flow rule, which is one of the central problems for plasticity theory, actually consists of two parts. First, there is the choice of the tensor measure of the rate of plastic deformation, several different forms of which have been advanced in the plasticity literature. Second, there is the choice of the particular flow function. Once the preferred measure has been selected, we argue that the flow function should be independent of the isochoric part of the plastic deformation. Finally, it is argued that the plastic stress power due to dislocation slip should also be independent of the isochoric part of the plastic deformation. This last assumption restricts the choices of the preferred measure of rate of plastic deformation.

Under some circumstances, it may be desirable to transform the internal variables to another equivalent set. Whereas some authors have argued that such transformations should be allowed to depend on current values of temperature and elastic strain or stress, we reject that possibility as being incompatible with the notion of instantaneous thermoelastic response. All acceptable transformations to a new set of internal variables, therefore, can depend only on the original set.

Section 5 closes with a discussion of the yield surface, which as usual is imagined as enclosing a region (or possibly only a single point) in stress space within which elastic, but not plastic, deformation can occur. Viscoplastic flow occurs only when the current stress lies outside the yield surface, with the rate of plastic deformation increasing with distance from the yield surface. In addition, we introduce the notion of structural surfaces. These are similar to the yield surface in that they are also imagined as surfaces in stress-temperature space, but each one encloses a region within which the corresponding internal variable cannot evolve further from its present state. These structural surfaces need not coincide with the yield surface, although they may as a special case. The notion of independent structural surfaces for the internal variables allows for the possibility of state changes that do not depend on the occurrence of plastic deformation, but which may actually occur in its absence. Examples of such a process might be static recovery and other relatively low temperature thermal treatments. Recognition of such possibilities requires that there can be no general relationship between increments of plastic strain and increments of internal variables, although there may be such a correlation in special cases. This has implications for the fraction of the rate of plastic work converted to heating.

The paper closes with Section 6, which contains a brief discussion of the completion of certain flow rules by the introduction of constitutive relations for plastic spin.

Although many of the results and viewpoints presented in this work are not new, it is often difficult to attribute them to any specific individual. Some of the papers with which we have significant points in common are Coleman and Gurtin (1967), Kratochvil (1971), Mandel (1974), Teodosiu and Sidoroff (1976), Anand (1985), and Cleja-Ţigoiu and Soós (1990). The paper by Cleja-Ţigoiu and Soós is particularly recommended for its extensive list of references and its critical evaluation of much of the early literature and some of the more recent literature. We share their viewpoint

that many of the erroneous interpretations in the plasticity literature, particularly those relating to the orientation of the intermediate (relaxed) configuration, are due to the failure to lay down an adequate constitutive framework prior to the discussion of such subtle issues. In this paper we propose a general framework for finite viscoplasticity, and in follow-up papers we will compare several classes of flow rules and examine the issues of invariance of the constitutive relations under rotations of the intermediate configuration or changes in the initial configuration.

2. Physical basis

A typical single crystal or polycrystalline grain deforms elastically under sufficiently small stresses or strains, but according to conventional views, the lattice eventually becomes unstable and dislocations begin to form. This is the beginning of plastic deformation by dislocation slip. As deformation continues, more dislocations form and existing dislocations move through the lattice. The net result is to add a certain amount of disorder or imperfection to the lattice. However, even at large plastic strains most of the material remains crystalline. For example, suppose that in a heavily deformed material there are $\mathcal{O}(10^{12}-10^{13})$ dislocation lines cutting through one square centimeter (Schaffer et al., 1995). This implies that the average space between dislocations is $\mathcal{O}(3-10)$ nm or approximately 10–40 atomic spacings or lattice parameters. Clearly most of the material must retain its crystalline form, and if it is true that stress arises from the distortion of the crystal lattice for elastic deformations, then it must still be true in a plastically deformed or plastically deforming solid. This observation is central to all further theoretical considerations.[1]

Since the presence of a dislocation distorts the lattice in nearby regions, elastic energy must be stored in its neighborhood. In fact, as is well known, a continuum representation of a dislocation results by considering a cut in the material, followed by a translation of one side of the cut surface relative to the other, and then followed by pasting the two sides of the surface back together again. Self-equilibrated stresses will then be concentrated around the edge of the cut, and again it is clear that elastic energy will be stored locally near the dislocation. Thus either the lattice or the continuum picture results in broadly similar conclusions. Point defects such as interstitial atoms or vacancies will also cause a local distortion in the lattice and will have a similar effect on the internal energy.

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[1] Although deformation twinning is also an important inelastic process in some materials, particularly those with few active slip systems, we do not specifically address twinning in this work. In principle the general evolution equations and flow rules considered in Section 5 could be applied to combined dislocation slip and twinning by associating some of the internal variables with the twinning process, but it is to be expected that a more realistic account of the relative effects of slip and twinning would result from incorporating some aspects of the kinematics of twinning (in a continuum sense) in the theoretical framework. In this regard the approach of Rajagopal and Srinivasa (1995, 1997, 1998a), based on the concept of multiple natural configurations, seems promising; see also Lapczyk et al. (1998). Variations of this concept have also been applied to phase transformations (Rajagopal and Srinivasa, 1998a, 1999).

To pursue this idea a bit further, imagine a crystalline solid within which "uniform plastic deformation" is occurring, that is to say, within which a distribution of dislocations forms and changes as plastic deformation develops. Consider a representative volume element $\mathcal{R}$ — a region small enough to be treated as a material point in a continuum model, yet large enough to be statistically representative of the local microstructure. According to our picture, there must be fluctuating stresses and strains within $\mathcal{R}$ so that on a fine enough scale, there is nothing really uniform about the deformation at all. However, suppose we decompose both the stress σ and elastic strain ε into an average over $\mathcal{R}$ plus a fluctuation. Then an increment of elastic work per unit current volume might be represented schematically as $\delta W = \frac{1}{V}\int_{\mathcal{R}}(\bar{\sigma} + \Delta\sigma) : \delta(\bar{\varepsilon} + \Delta\varepsilon)\mathrm{d}V$, where an overbar denotes the mean value over $\mathcal{R}$ (the reference and current configurations are taken to be identical here), and δ and Δ signify the increment and fluctuation, respectively. Since the average of the fluctuation within $\mathcal{R}$ is zero by definition, the increment of work may be written

$$\delta W = \bar{\sigma} : \delta\bar{\varepsilon} + \frac{1}{V}\int_{\mathcal{R}} \Delta\sigma : \delta(\Delta\varepsilon)\mathrm{d}V. \qquad (2.1)$$

The first term in (2.1) obviously represents the elastic work done by the average stress and strain increment, and the second term represents the extra elastic work done by the fluctuations. In the language of materials science, the first term may be associated with "long range forces," and the second term with "short range forces." To reiterate, the second term arises because of fluctuations in the elastic fields near dislocations or other disruptions in the lattice such as point defects, second phases, and grain boundaries. Alternatively, this term may be associated with the stored energy of cold work.

The precise meanings of the various terms in (2.1) have deliberately been left vague, but it seems clear that if plastic deformation occurs, some theoretical account of the energetic consequences, not only of the average fields but also of fluctuations around dislocations, should be given in any continuum model. In principle, it would seem possible to consider a plastically deformed solid to be simply a thermoelastic solid with special rules adjoined to allow for the formation and propagation of individual dislocations. In practice, however, if approached deterministically such a program is wildly unrealistic because of the vast numbers of dislocations that must be accounted for, although the large numbers would seem to be an advantage for a statistical approach. An alternate approach, and one that is in line with much other work in modern continuum mechanics, is to represent a plastically deformed material as being thermoelastic with respect to long range forces, as suggested above, and to represent the altered internal state by a finite set of internal variables. It is intended that these internal variables be associated conceptually with the fluctuations and their effects on the average response. They are a reflection of the altered microstructure produced by plastic deformation, so they may also be referred to as structural parameters. Two examples of internal variables, or structural parameters, that are commonly used in classical plasticity are a scalar work hardening parameter and a tensorial back stress.

Clarebrough et al. (1952, 1955, 1956, 1957, 1962) observed that plastic volume changes and changes in the stored energy of cold work occur simultaneously in several polycrystalline, fcc metals. Furthermore, Toupin and Rivlin (1960), using only second-order, nonlinear elasticity, derived formulas that describe dimensional changes in a crystal due to the presence of a dislocation, and Wright (1982) showed that these formulas and data, when supplemented by higher-order, known, elastic moduli, are fully consistent. Spitzig et al. (1975, 1976) measured permanent volume increases after plastic deformation in several steels. They found similar increases for tensile and compressive straining and also attributed the plastic volume change to the generation of dislocations. If plastic volume change due to the creation or annihilation of dislocations is to be taken into account, it should be regarded as a function of the internal variables characterizing the dislocation structure or as an internal variable itself. Relatively large plastic volume changes may result from the nucleation and growth of voids at sufficiently high tensile stresses. As is well-known, these processes can also be modeled by an internal state variable approach (Davison et al., 1977).

3. Multiplicative decomposition of the deformation gradient

3.1. The initial reference configuration[2]

Consider a crystalline body B, which initially occupies a stress-free configuration B_0 at uniform absolute temperature θ_0. The stress-free condition refers to the macroscopic stress in the continuum model. Microscopic residual stresses may exist, but their mean value is assumed to be negligible over any representative volume element in B_0. Let X denote a typical material point in this natural reference configuration, and let $x = x(X, t)$ denote the position of X at time $t \geqslant 0$. The deformation gradient relative to B_0 is $F = \partial x / \partial X$ and, since we assume that $\det F > 0$, the deformation gradient has the unique right and left polar decompositions

$$F = RU = VR. \tag{3.1}$$

Here, U and V are symmetric positive-definite tensors, called the right and left stretch tensors, respectively, and R is a rotation (i.e. a proper orthogonal tensor), called the local rotation tensor. The finite strain tensor E is defined by

$$E = \frac{1}{2}(C - I), \quad C = F^T F = U^2, \tag{3.2}$$

where I denotes the identity tensor and a superscript T denotes the transpose. The second Piola–Kirchhoff stress tensor $\tilde{T}_0$ relative to the initial configuration B_0 is defined in terms of the Cauchy stress tensor T by

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[2] The material in this subsection is standard; see Truesdell (1991), Ogden (1984), or Bowen 1989).

$$\tilde{T}_0 = (\det F)F^{-1}TF^{-T} = (\det U)U^{-1}(R^T TR)U^{-1}, \tag{3.3}$$

where a superscript $-T$ denotes the inverse of the transpose.

If $Q(t)$ is a time-dependent rotation, then the motion

$$x^*(X, t) = Q(t)x(X, t) \tag{3.4}$$

represents a rigid motion superposed on the motion $x = x(X, t)$. Since the motion $x^*(X, t)$ has deformation gradient

$$F^* = \frac{\partial x^*}{\partial X} = QF \tag{3.5}$$

relative to $\mathcal{B}_0$, the uniqueness of the stretch and rotation tensors U^*, V^*, and R^* in the polar decompositions of F^* imply that R, U, and V transform as

$$R^* = QR, \quad U^* = U, \quad V^* = QVQ^T \tag{3.6}$$

under the superposed rigid motion. The middle relation expresses the fact that U is invariant under superposed rigid motions. By (3.2) it follows that C and E are also invariant. The Cauchy stress tensor transforms as

$$T^* = QTQ^T. \tag{3.7}$$

This is regarded as a fundamental axiom of continuum mechanics, or it can be derived from a corresponding axiom for the contact force. Then the invariance of $\tilde{T}_0$ and $R^T TR$ under superposed rigid motions follows from (3.7) and the other relations above.

3.2. The local intermediate configuration

The total deformation in a plastically deforming solid is the result of thermoelastic lattice distortion and dislocation slip, together with any inelastic volume changes due to changing numbers of dislocations or to the nucleation and growth of voids. This suggests the usual multiplicative decomposition of the deformation gradient into an elastic part, F_e, and a plastic part, F_p[3]

$$F = F_e F_p. \tag{3.8}$$

F_e and F_p are assumed to have positive determinants, but unlike the total deformation gradient F, neither F_e nor F_p need be a gradient in itself. Nevertheless, it is

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[3] An alternative, reversed decomposition, $F = F_p F_e$, has been considered by Clifton (1972a), Nemat-Nasser (1979), and most recently by Lubarda (1999).

common to refer to $\mathbf{F_e}$ and $\mathbf{F_p}$ as the elastic and plastic deformation gradients, respectively. Since there are infinitely many ways to express a given tensor $\mathbf{F}$ as the product of two tensors, the elastic–plastic decomposition (3.8) is not a purely kinematical concept. For a given deformation history, the values of $\mathbf{F_e}$ and $\mathbf{F_p}$ in (3.8) must be determined by constitutive relations (see Sections 4 and 5). Nevertheless, certain qualitative properties may be laid down at this point, guided by the physical interpretation of these tensors.

The image of a neighborhood of a point $\mathbf{X}$ in $\mathcal{B}_0$ under the plastic deformation gradient $\mathbf{F_p}(\mathbf{X}, t)$ is often interpreted as a local, plastically deformed, intermediate configuration at the reference temperature θ_0. Then $\mathbf{F_e}(\mathbf{X}, t)$ is interpreted as a thermoelastic deformation of this intermediate configuration onto a neighborhood of $\mathbf{x}(\mathbf{X}, t)$ in the current configuration at temperature $\theta(\mathbf{X}, t)$. There are clearly some limitations to this interpretation since plastic deformation, elastic deformation, and temperature changes are generally occurring simultaneously. Nevertheless, this picture is conceptually useful and will be employed here.

Since, by assumption, the body occupies the reference configuration $\mathcal{B}_0$ at time $t = 0$, $\mathbf{F} = \mathbf{I}$ initially. And since $\mathbf{F_p}$ represents plastic deformation relative to this initial stress-free state, it is reasonable to assume that $\mathbf{F_p} = \mathbf{I}$ initially. Then by (3.8), we must also have $\mathbf{F_e} = \mathbf{I}$ initially. Thus

$$\mathbf{F}(\mathbf{X}, 0) = \mathbf{F_e}(\mathbf{X}, 0) = \mathbf{F_p}(\mathbf{X}, 0) = \mathbf{I}, \quad \forall \mathbf{X} \in \mathcal{B}_0. \tag{3.9}$$

It follows that the intermediate configuration initially coincides with the natural configuration $\mathcal{B}_0$.

The plastic deformation gradient can be decomposed into a dilatational part, $(\det \mathbf{F_p})^{1/3}$, representing any plastic volume change, and an isochoric part, $\bar{\mathbf{F}}_p$, representing dislocation slip,

$$\mathbf{F_p} = (\det \mathbf{F_p})^{1/3} \bar{\mathbf{F}}_p, \quad \det \bar{\mathbf{F}}_p = 1. \tag{3.10}$$

Let ρ, ρ_0, and ρ_R denote the mass densities in the current, initial, and intermediate configurations, respectively. Overall conservation of mass is expressed by

$$\rho \det \mathbf{F} = \rho_0. \tag{3.11}$$

Conservation of mass from the initial to the intermediate configuration and from the intermediate to the final configuration can be expressed by

$$\rho_R \det \mathbf{F_p} = \rho_0 \quad \text{and} \quad \rho \det \mathbf{F_e} = \rho_R. \tag{3.12}$$

Since $\det \mathbf{F} = \det \mathbf{F_e} \det \mathbf{F_p}$, these are completely consistent with (3.11). From (3.10) and (3.12) we have

$$\mathbf{F}_p = \left(\frac{\rho_0}{\rho_R}\right)^{1/3} \mathbf{\tilde{F}}_p. \tag{3.13}$$

It follows that $\rho_R = \rho_0$ and $\mathbf{F}_p = \mathbf{I}$ initially. The assumption of plastic incompressibility, which is not invoked here, would be expressed by $\rho_R \equiv \rho_0$ or, equivalently, by $\mathbf{F}_p \equiv \mathbf{\tilde{F}}_p$.

By analogy with $(3.3)_1$, the second Piola–Kirchhoff stress tensor $\mathbf{\tilde{T}}$ relative to the intermediate configuration is

$$\mathbf{\tilde{T}} = (\det\mathbf{F}_e)\mathbf{F}_e^{-1}\mathbf{T}\mathbf{F}_e^{-T} = \frac{\rho_R}{\rho}\mathbf{F}_e^{-1}\mathbf{T}\mathbf{F}_e^{-T}. \tag{3.14}$$

Since $\det\mathbf{F}_e > 0$, the elastic deformation gradient has unique polar decompositions

$$\mathbf{F}_e = \mathbf{R}_e\mathbf{U}_e = \mathbf{V}_e\mathbf{R}_e, \tag{3.15}$$

where $\mathbf{R}_e$ is the local elastic rotation tensor, and the symmetric positive-definite tensors $\mathbf{U}_e$ and $\mathbf{V}_e$ are the right and left elastic stretch tensors, respectively. By using $(3.15)_1$, we obtain an alternate expression for $\mathbf{\tilde{T}}$ analogous to the expression $(3.3)_2$ for $\mathbf{\tilde{T}}_0$,

$$\mathbf{\tilde{T}} = (\det\mathbf{U}_e)\mathbf{U}_e^{-1}(\mathbf{R}_e^T\mathbf{T}\mathbf{R}_e)\mathbf{U}_e^{-1}. \tag{3.16}$$

The local elastic rotation $\mathbf{R}_e$ maps the principal axes of $\mathbf{R}_e^T\mathbf{T}\mathbf{R}_e$ to those of $\mathbf{T}$; these two stress tensors have the same principal values, namely the principal Cauchy stresses.

For the superposed rigid motion $\mathbf{x}^* = \mathbf{Q}\mathbf{x}$, we have $\mathbf{F}_e^*\mathbf{F}_p^* = \mathbf{F}^* = \mathbf{Q}\mathbf{F} = \mathbf{Q}\mathbf{F}_e\mathbf{F}_p$. Thus, the transformation rules for $\mathbf{F}_e$ and $\mathbf{F}_p$ must be consistent with the condition

$$\mathbf{F}_e^*\mathbf{F}_p^* = \mathbf{Q}\mathbf{F}_e\mathbf{F}_p. \tag{3.17}$$

The assumption made here is the simplest one consistent with (3.17), namely, that under a superposed rigid motion, $\mathbf{F}_e$ and $\mathbf{F}_p$ transform as

$$\mathbf{F}_e^* = \mathbf{Q}\mathbf{F}_e \quad \text{and} \quad \mathbf{F}_p^* = \mathbf{F}_p. \tag{3.18}$$

In view of (3.17), either one of these rules implies the other. Note that $\mathbf{F}_e$ transforms like the total deformation gradient $\mathbf{F}$. The invariance of $\mathbf{F}_p$ is equivalent to the invariance of ρ_R and $\mathbf{F}_p$; therefore,

$$\rho_R^* = \rho_R, \quad \mathbf{F}_p^* = \mathbf{F}_p. \tag{3.19}$$

This is consistent with the view that a superposed rigid motion does not produce plastic volume change or dislocation slip. Additional motivation for the invariance of $\mathbf{F}_p$ is provided in Section 3.[4]

The elastic finite strain tensor $\mathbf{E}_e$ is defined by

$$\mathbf{E}_e = \frac{1}{2}(\mathbf{C}_e - \mathbf{I}), \quad \mathbf{C}_e = \mathbf{F}_e^T \mathbf{F}_e = \mathbf{U}_e^2, \tag{3.20}$$

which is analogous to the definition (3.2) of $\mathbf{E}$. From $(3.18)_1$ and the uniqueness of the stretch and rotation tensors in the polar decompostions of $\mathbf{F}_e$ and $\mathbf{F}_e^*$, we find that $\mathbf{R}_e$, $\mathbf{U}_e$, and $\mathbf{V}_e$ transform as

$$\mathbf{R}_e^* = \mathbf{Q}\mathbf{R}_e, \quad \mathbf{U}_e^* = \mathbf{U}_e, \quad \mathbf{V}_e^* = \mathbf{Q}\mathbf{V}_e\mathbf{Q}^T \tag{3.21}$$

under a superposed rigid motion. In addition to $\mathbf{U}_e$, the tensors $\mathbf{C}_e$, $\mathbf{E}_e$, $\tilde{\mathbf{T}}$, and $\mathbf{R}_e^T \mathbf{T} \mathbf{R}_e$ are also invariant under a superposed rigid motion. These transformation rules for tensors defined relative to the intermediate configuration are analogous to the transformation rules in Section 3.1 for tensors defined relative to the initial configuration.

3.3. The small elastic shear strain approximation

From $(3.12)_2$ and the polar decomposition of $\mathbf{F}_e$, we have

$$\det\mathbf{F}_e = \det\mathbf{U}_e = \det\mathbf{V}_e = \frac{\rho_R}{\rho}. \tag{3.22}$$

The elastic deformation gradient $\mathbf{F}_e$ may be decomposed into a dilatational part, $(\det\mathbf{F}_e)^{1/3}$, and an isochoric part, $\bar{\mathbf{F}}_e$,

$$\mathbf{F}_e = (\det\mathbf{F}_e)^{1/3}\bar{\mathbf{F}}_e = \left(\frac{\rho_R}{\rho}\right)^{1/3}\bar{\mathbf{F}}_e, \quad \det\bar{\mathbf{F}}_e = 1. \tag{3.23}$$

Similarly, the elastic stretch tensors may be decomposed into dilatational and isochoric parts,

$$\mathbf{U}_e = \left(\frac{\rho_R}{\rho}\right)^{1/3}\bar{\mathbf{U}}_e, \quad \mathbf{V}_e = \left(\frac{\rho_R}{\rho}\right)^{1/3}\bar{\mathbf{V}}_e, \quad \det\bar{\mathbf{U}}_e = \det\bar{\mathbf{V}}_e = 1. \tag{3.24}$$

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[4] Some authors adopt the transformation rules $\mathbf{F}_e^* = \mathbf{Q}\mathbf{F}_e\mathbf{Q}^T$ and $\mathbf{F}_p^* = \mathbf{Q}\mathbf{F}_p$. Indeed, these would seem to be the only other simple rules consistent with (3.17). However, the rule $\mathbf{F}_p^* = \mathbf{Q}\mathbf{F}_p$ is inconsistent with the property of instantaneous elastic response introduced in Section 4.4.This transformation rule also has the undesirable property that a rigid motion superposed on the initial undeformed state (where $\mathbf{F}_p = \mathbf{I}$) results in $\mathbf{F}_p^* = \mathbf{Q}$. We do not consider a pure rotation to be a plastic deformation in the sense of crystallographic slip.

Then $\mathbf{F_e}$ has the polar decomposition

$$\mathbf{F_e} = \mathbf{R_e}\mathbf{U_e} = \mathbf{V_e}\mathbf{R_e},\tag{3.25}$$

and

$$\mathbf{C_e} = \left(\frac{\rho_R}{\rho}\right)^{2/3}\mathbf{C_e}, \quad \mathbf{C_e} = \mathbf{F_e^T}\mathbf{F_e} = U_e^2, \quad \det\mathbf{C_e} = 1.\tag{3.26}$$

The symmetric positive-definite tensors $\mathbf{U_e}$ and $\mathbf{V_e}$ and their eigenvalues $\bar{\lambda}_1^e$, $\bar{\lambda}_2^e$, $\bar{\lambda}_3^e$ are independent of the dilatational and rotational parts of $\mathbf{F_e}$ and thus are measures of elastic distortion only. It follows that $\mathbf{U_e} - \mathbf{I}$ and $\mathbf{V_e} - \mathbf{I}$ can be regarded as tensor measures of elastic shear strain and that

$$\|\mathbf{U_e} - \mathbf{I}\| = \|\mathbf{V_e} - \mathbf{I}\| = \left[\sum_{i=1}^{3}\left(\bar{\lambda}_i^e - 1\right)^2\right]^{1/2}\tag{3.27}$$

measures the magnitude of the elastic shear strain, independent of the elastic volumetric strain and the elastic rotation.

Plastic flow limits the elastic shear strains that a material can support, and for metals $\|\mathbf{U_e} - \mathbf{I}\|$ is small compared to unity. Then $\|\mathbf{U_e} - \mathbf{I}\|^2$ is small compared to $\|\mathbf{U_e} - \mathbf{I}\|$, and so on. By a "small elastic shear strain approximation" we mean any approximation obtained by neglecting these relatively small terms. For example, since $\mathbf{U_e} = \mathbf{I} + (\mathbf{U_e} - \mathbf{I})$ and since we may neglect the term $\mathbf{U_e} - \mathbf{I}$ relative to $\mathbf{I}$ for small elastic shear strains, on using $(3.24)_1$ we obtain the approximations

$$\mathbf{U_e}\mathbf{H} \approx \mathbf{H}\mathbf{U_e} \approx \left(\frac{\rho_R}{\rho}\right)^{1/3}\mathbf{H}\tag{3.28}$$

for any tensor $\mathbf{H}$. Since $\|\mathbf{U_e^{-1}} - \mathbf{I}\|$ and $\|\mathbf{C_e} - \mathbf{I}\|$ are small compared to unity when $\|\mathbf{U_e} - \mathbf{I}\|$ is small compared to unity, we also have

$$\mathbf{U_e^{-1}}\mathbf{H} \approx \mathbf{H}\mathbf{U_e^{-1}} \approx \left(\frac{\rho_R}{\rho}\right)^{-1/3}\mathbf{H} \quad \text{and} \quad \mathbf{C_e}\mathbf{H} \approx \mathbf{H}\mathbf{C_e} \approx \left(\frac{\rho_R}{\rho}\right)^{2/3}\mathbf{H}\tag{3.29}$$

for any tensor $\mathbf{H}$. From $(3.29)_1$, (3.22), and the exact expression (3.16) for $\tilde{\mathbf{T}}$, we have

$$\tilde{\mathbf{T}} \approx \left(\frac{\rho_R}{\rho}\right)^{1/3}\mathbf{R_e^T}\mathbf{T}\mathbf{R_e}\tag{3.30}$$

in the small elastic shear strain approximation. Similarly, (3.28), $(3.24)_1$, and the polar decomposition $\mathbf{F_e} = \mathbf{R_e}\mathbf{U_e}$ yield

$$\mathbf{F}_e^T \mathbf{H} \mathbf{F}_e \approx \left(\frac{\rho_R}{\rho}\right)^{2/3} \mathbf{R}_e^T \mathbf{H} \mathbf{R}_e \tag{3.31}$$

for any tensor **H**.

Although the small elastic shear strain approximation is sufficiently accurate for some problems in metal plasticity, there are circumstances where higher order terms need to be retained.[5] For this reason we develop the general theory for finite elastic strains and then occasionally point out the simplifications that result for small elastic shear strains.

4. Thermodynamic restrictions

4.1. Balance laws and the entropy inequality[6]

The local form of balance of linear momentum in the current configuration is

$$\text{div}\mathbf{T} + \rho\mathbf{b} = \rho\dot{\mathbf{v}}, \tag{4.1}$$

where **b** is the body force per unit mass, **v** is the particle velocity, and a superposed dot denotes the material time derivative. After balance of mass and momentum are taken into account, the local form of balance of energy in the current configuration reduces to

$$\dot{e} = \mathcal{P} - \frac{\text{div }\mathbf{q}}{\rho} + r, \tag{4.2}$$

where e is the internal energy per unit mass, **q** is the heat flux vector, and r is the rate of supply of energy per unit mass from the exterior of the body. The stress power per unit mass, $\mathcal{P}$, is given by

$$\mathcal{P} = \frac{\mathbf{T}:\mathbf{L}}{\rho} = \frac{\mathbf{T}:\mathbf{D}}{\rho} = \frac{\tilde{\mathbf{T}}_0:\dot{\mathbf{E}}}{\rho_0}. \tag{4.3}$$

Here **L** is the spatial velocity gradient and the stretching tensor **D** is the symmetric part of **L**,

$$\nabla\mathbf{v} = \mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \quad \mathbf{D} = \text{sym}\mathbf{L} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T). \tag{4.4}$$

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[5] For example, see Clifton (1972b) and Herrmann (1976) for discussions of nonlinear elastic effects on the decay of the elastic precursor in elastic–plastic waves generated by plate impact.

[6] The results in this subsection are standard; see Truesdell (1984) or Bowen (1989).

The last expression in (4.3) follows from $(3.3)_1$, (3.11), and the first of the relations

$$\dot{\mathbf{E}} = \mathbf{F}^T \mathbf{DF} = \text{sym}\left(\mathbf{F}^T \dot{\mathbf{F}}\right) = \frac{1}{2}\dot{\mathbf{C}} = \frac{1}{2}\left(\mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}\right). \tag{4.5}$$

Conservation of mass in the current configuration is expressed by

$$-\frac{\dot{\rho}}{\rho} = \frac{(\det \mathbf{F})^{\cdot}}{\det \mathbf{F}} = \text{tr}\left(\dot{\mathbf{F}}\mathbf{F}^{-1}\right) = \text{tr}\mathbf{L} = \text{tr}\mathbf{D}. \tag{4.6}$$

In continuum mechanics, the second law of thermodynamics is often expressed by the Clausius–Duhem inequality, hereafter simply referred to as the entropy inequality. The local form of this inequality in the current configuration is

$$\dot{\eta} \geqslant -\frac{1}{\rho}\text{div}\left(\frac{\mathbf{q}}{\theta}\right) + \frac{r}{\theta}, \tag{4.7}$$

where η is the entropy per unit mass. The internal dissipation per unit mass, δ, is defined as the amount by which $\theta\dot{\eta}$ exceeds the local heating:

$$\delta = \theta\dot{\eta} - \left(-\frac{\text{div }\mathbf{q}}{\rho} + r\right). \tag{4.8}$$

By expanding the divergence term in (4.7) and using (4.8), the entropy inequality may be written as

$$\delta - \frac{\mathbf{q}\cdot\nabla\theta}{\rho\theta} \geqslant 0. \tag{4.9}$$

In view of (4.8), the energy balance Eq. (4.2) is equivalent to

$$\delta = \mathcal{P} + \theta\dot{\eta} - \dot{e}. \tag{4.10}$$

Alternatively, one may define the internal dissipation by (4.10). Then (4.8) expresses the balance of energy, which is usually written as

$$\theta\dot{\eta} = \delta - \frac{\text{div }\mathbf{q}}{\rho} + r. \tag{4.11}$$

If ψ denotes the Helmholz free energy per unit mass,

$$\psi = e - \theta\eta, \tag{4.12}$$

then the internal dissipation may be expressed as

$$\delta = \mathcal{P} - \left(\dot{\psi} + \eta\dot{\theta}\right). \tag{4.13}$$

4.2. Elastic–plastic decomposition of the stress power

By taking the material time derivative of the multiplicative decompostion $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ and using (4.4) and (3.20), we obtain additive decompositions of the velocity gradient and the stretching tensor into elastic and plastic parts:

$$\mathbf{L} = \mathbf{L}_e + \mathbf{L}_p, \quad \mathbf{L}_e = \dot{\mathbf{F}}_e\mathbf{F}_e^{-1}, \quad \mathbf{L}_p = \mathbf{F}_e\left(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}\right)\mathbf{F}_e^{-1}, \tag{4.14}$$

$$\mathbf{D} = \mathbf{D}_e + \mathbf{D}_p, \quad \mathbf{D}_e = \mathrm{sym}\mathbf{L}_e = \mathbf{F}_e^{-T}\dot{\mathbf{E}}_e\mathbf{F}_e^{-1}, \quad \mathbf{D}_p = \mathrm{sym}\mathbf{L}_p. \tag{4.15}$$

Note that just as (4.6) expresses conservation of mass in the current configuration, the above relations and (3.12)$_1$ imply

$$-\frac{\dot{\rho}_R}{\rho_R} = \frac{(\det\mathbf{F}_p)^{\cdot}}{\det\mathbf{F}_p} = \mathrm{tr}\left(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}\right) = \mathrm{tr}\mathbf{L}_p = \mathrm{tr}\mathbf{D}_p, \tag{4.16}$$

which is an expression of conservation of mass in the intermediate configuration.

From (4.14), (4.15), and (4.3), we see that the multiplicative decomposition of the deformation gradient leads to an additive decomposition of the total stress power (per unit mass) into elastic and plastic parts,

$$\mathcal{P} = \mathcal{P}_e + \mathcal{P}_p, \tag{4.17}$$

with

$$\rho\mathcal{P}_e = \mathbf{T} : \mathbf{L}_e = \mathbf{T} : \mathbf{D}_e, \quad \rho\mathcal{P}_p = \mathbf{T} : \mathbf{L}_p = \mathbf{T} : \mathbf{D}_p, \tag{4.18}$$

On using (4.10), (4.13), and the elastic-plastic decomposition of the stress power, the internal dissipation δ may be expressed as

$$\delta = \mathcal{P}_p - \dot{\mathcal{W}}_s, \tag{4.19}$$

where

$$\dot{\mathcal{W}}_s = \dot{\psi} + \eta\dot{\theta} - \mathcal{P}_e = \dot{e} - \theta\dot{\eta} - \mathcal{P}_e. \tag{4.20}$$

Given the constitutive assumptions introduced in Sections 4.3–4.4, it will be shown in Section 4.6 that $\dot{\mathcal{W}}_s$ may be associated with the rate of change of the stored energy of cold work. In view of (4.19), the entropy inequality (4.9) may be written as

$$\mathcal{P}_p - \dot{\mathcal{W}}_s - \frac{\mathbf{q} \cdot \nabla\theta}{\rho\theta} \geqslant 0. \tag{4.21}$$

Alternate tensor measures of total, elastic, and plastic strain rate are obtained by transforming the tensors in (4.14)–(4.15) back to the intermediate configuration:

$$\mathbf{F}_e^T \mathbf{L} \mathbf{F}_e \equiv \tilde{\mathbf{L}} = \tilde{\mathbf{L}}_e + \tilde{\mathbf{L}}_p, \tag{4.22}$$

where

$$\tilde{\mathbf{L}}_e \equiv \mathbf{F}_e^T \mathbf{L}_e \mathbf{F}_e = \mathbf{F}_e^T \dot{\mathbf{F}}_e, \tag{4.23}$$

$$\tilde{\mathbf{L}}_p \equiv \mathbf{F}_e^T \mathbf{L}_p \mathbf{F}_e = \mathbf{C}_e \dot{\mathbf{F}}_p \mathbf{F}_p^{-1}; \tag{4.24}$$

and

$$\text{sym}\tilde{\mathbf{L}} = \mathbf{F}_e^T \mathbf{D} \mathbf{F}_e \equiv \tilde{\mathbf{D}} = \dot{\tilde{\mathbf{E}}}_e + \tilde{\mathbf{D}}_p, \tag{4.25}$$

where

$$\dot{\tilde{\mathbf{E}}}_e = \mathbf{F}_e^T \mathbf{D}_e \mathbf{F}_e = \text{sym}\tilde{\mathbf{L}}_e, \tag{4.26}$$

$$\tilde{\mathbf{D}}_p \equiv \mathbf{F}_e^T \mathbf{D}_p \mathbf{F}_e = \text{sym}\tilde{\mathbf{L}}_p = \text{sym}\left(\mathbf{C}_e \dot{\mathbf{F}}_p \mathbf{F}_p^{-1}\right). \tag{4.27}$$

From (4.25) it follows that $\dot{\tilde{\mathbf{E}}}_e$ and $\tilde{\mathbf{D}}_p$ can be regarded as the elastic and plastic parts of the total strain rate $\tilde{\mathbf{D}}$. Note that $\tilde{\mathbf{D}}$ can also be obtained by transforming the total strain rate $\dot{\mathbf{E}}$ from the initial configuration to the intermediate configuration,

$$\tilde{\mathbf{D}} = \mathbf{F}_p^{-T} \dot{\mathbf{E}} \mathbf{F}_p^{-1}. \tag{4.28}$$

The tensors $\tilde{\mathbf{L}}$, $\tilde{\mathbf{L}}_e$, $\tilde{\mathbf{L}}_p$, $\tilde{\mathbf{D}}$, $\dot{\tilde{\mathbf{E}}}_e$, and $\tilde{\mathbf{D}}_p$ are invariant under superposed rigid motions.

From (3.14), (4.17) and (4.18), and the relations above, we obtain alternate expressions for the total, the elastic, and the plastic stress power (per unit mass) in terms of quantities in the intermediate configuration:

$$\rho_R \mathcal{P} = \tilde{\mathbf{T}} : \tilde{\mathbf{L}} = \tilde{\mathbf{T}} : \tilde{\mathbf{D}}, \tag{4.29}$$

$$\rho_R \mathcal{P}_e = \tilde{\mathbf{T}} : \tilde{\mathbf{L}}_e = \tilde{\mathbf{T}} : \dot{\tilde{\mathbf{E}}}_e, \quad \rho_R \mathcal{P}_p = \tilde{\mathbf{T}} : \tilde{\mathbf{L}}_p = \tilde{\mathbf{T}} : \tilde{\mathbf{D}}_p. \tag{4.30}$$

In view of (4.30), the stress tensor $\tilde{\mathbf{T}}$ is said to be (work) conjugate to the elastic strain tensor $\mathbf{E}_e$ and the plastic strain rate tensor $\tilde{\mathbf{D}}_p$. This concept is generalized to

other stress and strain measures in Section 4.10. The plastic strain rate tensor $\tilde{\mathbf{D}}_p$ has been emphasized by several authors; see Miehe (1994), Maugin (1994), Cleja-Ţigoiu and Maugin (2000), and the references therein. Decomposition of the plastic stress power and the plastic strain rates $\tilde{\mathbf{D}}_p$ and $\mathbf{D}_p$ into dilatational and shearing parts is discussed in Section 4.8. From (4.30) and the expression (4.24) for $\tilde{\mathbf{L}}_p$, the plastic stress power may also be expressed as

$$\rho_R \mathcal{P}_p = \mathbf{C}_e \tilde{\mathbf{T}} : \dot{\mathbf{F}}_p \mathbf{F}_p^{-1}. \tag{4.31}$$

In view of this result, some authors work with the stress tensor $\mathbf{C}_e \tilde{\mathbf{T}} = (\rho_R/\rho) \mathbf{F}_e^T \mathbf{T} \mathbf{F}_e^{-T}$, which is generally not symmetric (Teodosiu and Sidoroff, 1976; Cleja-Ţigoiu and Maugin, 2000).

For small elastic shear strains, (3.31) and (3.29)$_2$ yield approximations for the rates in (4.22)–(4.27). For example,

$$\left(\frac{\rho_R}{\rho}\right)^{-2/3} \tilde{\mathbf{L}}_p \approx \mathbf{R}_e^T \mathbf{L}_p \mathbf{R}_e \approx \dot{\mathbf{F}}_p \mathbf{F}_p^{-1} \tag{4.32}$$

and

$$\left(\frac{\rho_R}{\rho}\right)^{-2/3} \tilde{\mathbf{D}}_p \approx \mathbf{R}_e^T \mathbf{D}_p \mathbf{R}_e. \tag{4.33}$$

Note that, while (4.27) and (3.29)$_2$ might suggest the approximation $\tilde{\mathbf{D}}_p \approx \text{sym}\left(\dot{\mathbf{F}}_p \mathbf{F}_p^{-1}\right)$, this approximation is not valid without some restrictions on the relative magnitude of $\text{skw}\left(\dot{\mathbf{F}}_p \mathbf{F}_p^{-1}\right)$ and $\text{sym}\left(\dot{\mathbf{F}}_p \mathbf{F}_p^{-1}\right)$. This point is discussed in more detail in a follow-up paper (Scheidler and Wright, 2001).

4.3. Thermoelastic constitutive assumptions

To reflect the ideas expressed in Section 2, we assume that the internal energy e, the free energy ψ, and the Cauchy stress $\mathbf{T}$ depend only on the current values of the elastic part of the deformation gradient, the absolute temperature, and the internal variables. Let $q_1, \ldots, q_N$ and $\mathbf{A}_1, \ldots, \mathbf{A}_K$ denote the scalar and tensor internal variables, respectively. Then

$$\psi = \hat{\psi}(\mathbf{F}_e, \theta, q_1, \ldots, q_N, \mathbf{A}_1, \ldots, \mathbf{A}_K), \tag{4.34}$$

$$e = \hat{e}(\mathbf{F}_e, \theta, q_1, \ldots, q_N, \mathbf{A}_1, \ldots, \mathbf{A}_K), \tag{4.35}$$

$$\mathbf{T} = \hat{\tau}(\mathbf{F}_e, \theta, q_1, \ldots, q_N, \mathbf{A}_1, \ldots, \mathbf{A}_K), \tag{4.36}$$

for some smooth[7] functions $\hat{\psi}$, $\hat{e}$, and $\hat{\tau}$. Definitions (4.12) and (3.14) imply that η and $\tilde{T}$ also exhibit the same functional dependencies.

Although the general framework developed in this paper is not restricted to tensor internal variables of second order, for simplicity we regard the A_k as second-order tensors since specific examples of higher-order tensors are not discussed. We do not require that the A_k be symmetric. The list of internal variables is assumed to be functionally independent in the sense that no internal variable can be expressed as a function of the others. Depending on the context, we use q_n and A_k to denote either typical scalar and tensor internal variables or the lists of these variables. For example, the constitutive relation (4.34) is abbreviated as $\psi = \hat{\psi}(F_e, \theta, q_n, A_k)$.

Observe that, while F_e is measured relative to a changing intermediate configuration, the response functions in (4.34)–(4.36) are fixed. For a single crystal this assumption is justified if F_p evolves in such a way that the orientation of the crystal lattice in the intermediate configuration coincides with the lattice orientation in the initial configuration. Then the thermoelastic response relative to the intermediate configuration should coincide, at least approximately, with the initial thermoelastic response, any differences being due to the altered microstructure as expressed by changes in the internal variables. Following Mandel (1974), such an intermediate configuration is often referred to as isoclinic. Since an isoclinic intermediate configuration is clearly invariant under a superposed rigid motion, so is the plastic deformation gradient F_p that defines it (Kratochvil, 1971). Thus the assumed invariance of F_p in Section 3.2 is consistent with the above interpretation. For a polycrystal, F_e and F_p represent average elastic and plastic deformations over the individual grains in a representative volume element. Since these grains may rotate relative to one another, the concept of an isoclinic intermediate configuration does not extend from single crystals to polycrystals. Nevertheless, the *existence* of an invariant intermediate configuration relative to which the thermoelastic response can be approximated by relations of the general form (4.34)–(4.36) does not seem unreasonable. For a randomly oriented polycrystal, the lack of *uniqueness* of an intermediate configuration with these properties has led to much debate and some confusion in the literature.

The dependence of the stress on some or all of the internal variables may be negligible for certain problems. For example, even large changes in a scalar work hardening parameter would be expected to have at most a small influence on the elastic response. However, the effects of evolving microstructure on the stress cannot always be neglected. Two cases come to mind immediately. First, for sufficiently large deformations of initially isotropic polycrystals, the development of texture due to reorientation of the crystal lattices of grains can result in significant anisotropy in both the incremental plastic and elastic responses. Second, when void growth occurs, the effective elastic moduli decrease with increasing void volume fraction.

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[7] Henceforth, all constitutive functions are assumed to be smooth unless stated otherwise. If the body is inhomogeneous then (4.34)–(4.36) should include an explicit dependence on X. Although we allow for this possibility, the argument X is suppressed for simplicity.

Since the initial configuration is unstressed, (4.36) and (3.9) imply that $\hat{\tau}(\mathbf{I}, \theta_0, q_n^0, \mathbf{A}_k^0) = \mathbf{0}$, where q_n^0 and $\mathbf{A}_k^0$ are the initial values of the internal variables. If the current configuration coincides with the intermediate configuration, so that $\mathbf{F}_e = \mathbf{I}$, and if $\theta = \theta_0$, then $\mathbf{T} = \hat{\tau}(\mathbf{I}, \theta_0, q_n, \mathbf{A}_k)$. It follows that the intermediate configuration is necessarily stress-free if the internal variables have not changed from their initial values or if the dependence of the stress on the internal variables can be neglected. We will assume that

$$\hat{\tau}(\mathbf{I}, \theta_0, q_n, \mathbf{A}_k) = \mathbf{0} \tag{4.37}$$

for all possible values of the internal variables, which implies that the intermediate configuration is always a stress-free state.[8] For this reason it is often referred to as a relaxed configuration. This assumption is conventional and seems to be consistent with the physical interpretations discussed above. However, it does not imply that the intermediate configuration at time t can necessarily be attained by unloading from the current configuration at time t, since the process of unloading may result in further plastic deformation and hence in a new intermediate configuration.

The scalar internal variables, like all true scalar fields, are assumed to be invariant under superposed rigid motions. The transformation properties of the tensor internal variables depend on their physical interpretation. For example, if we regarded $\mathbf{A}_1$ as a Cauchy-type back stress, then it should transform like the Cauchy stress tensor. Instead, we assume that the tensor internal variables $\mathbf{A}_k$, like $\tilde{\mathbf{T}}$ and $\mathbf{E}_e$, are invariant under superposed rigid motions. Thus

$$q_n^* = q_n, \quad \mathbf{A}_k^* = \mathbf{A}_k, \tag{4.38}$$

and the tensors $\mathbf{A}_k$ may be thought of as residing in the intermediate configuration.

At this stage, the assumption $\mathbf{A}_k^* = \mathbf{A}_k$ is not really a constitutive restriction since for each of the tensor internal variables $\mathbf{A}_k$, we may define a corresponding tensor internal variable $\mathbf{B}_k$ that does transform like the Cauchy stress tensor, that is,

$$\mathbf{B}_k^* = \mathbf{Q}\mathbf{B}_k\mathbf{Q}^T \tag{4.39}$$

under a superposed rigid motion $\mathbf{x}^* = \mathbf{Q}\mathbf{x}$. For example, each of the choices

$$\mathbf{B}_k = \mathbf{R}_e\mathbf{A}_k\mathbf{R}_e^T, \mathbf{F}_e\mathbf{A}_k\mathbf{F}_e^T, \quad \text{or} \quad \mathbf{F}_e^{-T}\mathbf{A}_k\mathbf{F}_e^{-1} \tag{4.40}$$

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[8] This assumption is used only occasionally in the rest of the paper, so it could be weakened or even dropped. In particular, the derivations of the thermodynamic restrictions in Sections 4.5–4.7 make no use of (4.37). For some materials and processes it may be more reasonable to assume that $\hat{\tau}(\mathbf{I}, \theta_R, q_n, \mathbf{A}_k) = \mathbf{0}$ for some temperature $\theta_R = \bar{\theta}_R(q_n, \mathbf{A}_k)$. Then the intermediate configuration may still be interpreted as stress-free provided that the temperature in the intermediate configuration is taken to be θ_R instead of θ_0. For example, in an ostensibly isotropic polycrystalline material with less than cubic symmetry in the fundamental crystals, either pressure or thermal cycling may cause plastic deformation at crystal boundaries to maintain continuity. Such changes must be manifested in the internal variables and could require a change of reference temperature in the manner suggested.

satisfies this condition. On solving (4.40) for $\mathbf{A}_k$, we obtain

$$\mathbf{A}_k = \mathbf{R}_e^T \mathbf{B}_k \mathbf{R}_e, \quad \mathbf{F}_e^{-1} \mathbf{B}_k \mathbf{F}_e^{-T}, \quad \text{or} \quad \mathbf{F}_e^T \mathbf{B}_k \mathbf{F}_e. \tag{4.41}$$

On substituting (4.41) into the constitutive relation $\psi = \hat{\psi}(\mathbf{F}_e, \theta, q_n, \mathbf{A}_k)$, we obtain a relation of the form

$$\psi = \hat{\Psi}(\mathbf{F}_e, \theta, q_n, \mathbf{B}_k) \tag{4.42}$$

for some function $\hat{\Psi}$, which depends on the particular relations between the $\mathbf{A}_k$ and the $\mathbf{B}_k$. Conversely, if ψ is assumed to satisfy a constitutive relation of the form (4.42) and the tensor internal variables $\mathbf{B}_k$ transforms as in (4.39), then (4.41) defines corresponding tensor internal variables $\mathbf{A}_k$ that are invariant under superposed rigid motions; and on solving (4.41) for $\mathbf{B}_k$, we obtain (4.40), which yields a constitutive relation of the form $\psi = \hat{\psi}(\mathbf{F}_e, \theta, q_n, \mathbf{A}_k)$ when substituted into (4.42). Note that for the relations above or for variations such as $\mathbf{A}_k = (\det\mathbf{F}_e)\mathbf{F}_e^{-1}\mathbf{B}_k\mathbf{F}_e^{-T}$, the tensor $\mathbf{A}_k$ is symmetric iff $\mathbf{B}_k$ is symmetric. Also note that in the small elastic shear strain approximation, the second and third choices in (4.40) differ from $\mathbf{R}_e\mathbf{A}_k\mathbf{R}_e^T$ by the factors $(\rho_R/\rho)^{2/3}$ and $(\rho_R/\rho)^{-2/3}$, respectively.

Since ψ, θ, q_n, and $\mathbf{A}_k$ are invariant under a superposed rigid motion $\mathbf{x}^* = \mathbf{Q}\mathbf{x}$, whereas $\mathbf{F}_e^* = \mathbf{Q}\mathbf{F}_e$, it follows that

$$\psi = \hat{\psi}(\mathbf{F}_e, \theta, q_n, \mathbf{A}_k) = \hat{\psi}(\mathbf{Q}\mathbf{F}_e, \theta, q_n, \mathbf{A}_k) = \hat{\psi}(\mathbf{U}_e, \theta, q_n, \mathbf{A}_k) \tag{4.43}$$

for every rotation $\mathbf{Q}$, where the last relation follows from the polar decomposition $\mathbf{F}_e = \mathbf{R}_e\mathbf{U}_e$ by taking $\mathbf{Q} = \mathbf{R}_e^T$. Since

$$\mathbf{U}_e = \sqrt{\mathbf{I} + 2\mathbf{E}_e}, \tag{4.44}$$

(4.43) is equivalent to the reduced form

$$\psi = \tilde{\psi}(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k) \tag{4.45}$$

for some smooth function $\tilde{\psi}$. Results analogous to (4.43) and (4.45) hold for e, η, and $\tilde{\mathbf{T}}$, since these variables are also invariant under superposed rigid motions. Thus

$$e = \tilde{e}(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k), \tag{4.46}$$

$$\eta = \tilde{\eta}(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k), \tag{4.47}$$

$$\tilde{\mathbf{T}} = \tilde{\tau}(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k), \tag{4.48}$$

and (4.37) implies that

$$\tilde{\tau}(\mathbf{0}, \theta_0, q_n.\mathbf{A}_k) = \mathbf{0}. \tag{4.49}$$

Constitutive relations such as the reduced forms (4.45)–(4.48), for which the dependent variables transform properly under superposed rigid motions, are said to be properly invariant or objective.[9] By (3.14),

$$\mathbf{T} = (\det\mathbf{F}_e)^{-1}\mathbf{F}_e\tilde{\mathbf{T}}\mathbf{F}_e^{\mathrm{T}} = \frac{\rho}{\rho_R}\mathbf{F}_e\tilde{\mathbf{T}}\mathbf{F}_e^{\mathrm{T}}. \tag{4.50}$$

Then (4.50) with $\tilde{\mathbf{T}}$ given by (4.48) is a properly invariant constitutive relation for the Cauchy stress. Properly invariant constitutive relations in terms of tensor internal variables $\mathbf{B}_k$ that transform like the Cauchy stress are given by (4.45)–(4.48) with $\mathbf{A}_k$ replaced by any of the expressions in (4.41).

From Eq. (4.45), the material time derivative of the free energy is given by

$$\dot{\psi} = \frac{\partial\psi}{\partial\mathbf{E}_e} : \dot{\mathbf{E}}_e + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \sum_{n=1}^{N}\frac{\partial\psi}{\partial q_n}\dot{q}_n + \sum_{k=1}^{M}\frac{\partial\psi}{\partial\mathbf{A}_k} : \dot{\mathbf{A}}_k. \tag{4.51}$$

Then from (4.20)$_1$ and the expression (4.30) for the elastic part of the stress power,

$$\dot{\mathcal{W}}_s = \left(\frac{\partial\psi}{\partial\mathbf{E}_e} - \frac{\tilde{\mathbf{T}}}{\rho_R}\right) : \dot{\mathbf{E}}_e + \left(\frac{\partial\psi}{\partial\theta} + \eta\right)\dot{\theta} + \sum_{n=1}^{N}\frac{\partial\psi}{\partial q_n}\dot{q}_n + \sum_{k=1}^{K}\frac{\partial\psi}{\partial\mathbf{A}_k} : \dot{\mathbf{A}}_k. \tag{4.52}$$

On using the above expression for $\dot{\mathcal{W}}_s$, the entropy inequality (4.21) becomes

$$\left(\frac{\partial\psi}{\partial\mathbf{E}_e} - \frac{\tilde{\mathbf{T}}}{\rho_R}\right) : \dot{\mathbf{E}}_e + \left(\frac{\partial\psi}{\partial\theta} + \eta\right)\dot{\theta} \leqslant \left(\mathcal{P}_p - \sum_{n=1}^{N}\frac{\partial\psi}{\partial q_n}\dot{q}_n - \sum_{k=1}^{K}\frac{\partial\psi}{\partial\mathbf{A}_k} : \dot{\mathbf{A}}_k\right) - \frac{\mathbf{q}\cdot\nabla\theta}{\rho\theta}.$$
$$\tag{4.53}$$

The next step is to determine the restrictions imposed by the entropy inequality (4.53) on the constitutive functions $\tilde{\psi}$, $\tilde{e}$, $\tilde{\tau}$, and $\tilde{\eta}$ in (4.45)–(4.48). These restrictions depend on evolution properties of the internal state variables and the plastic deformation gradient that have yet to be specified. The property of instantaneous thermoelastic response, which is introduced in the next subsection, implies that the expressions in parentheses on the left-hand side of the entropy inequality (4.53) must be zero. Consequently, we obtain the classical thermoelastic potentials as well as a plastic dissipation inequality and a restriction on the plastic volume change.

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[9] If $\mathbf{x}^* = \mathbf{Q}\mathbf{x}$ is interpreted as a change of frame (or observer), and if the transformation rules for the field variables are interpreted accordingly, then identical results follow from the axiom that constitutive relations be independent of the frame of reference, in which case the reduced forms (4.45)–(4.48) are referred to as frame-indifferent; see Truesdell (1991) or Ogden (1984).

The use of the Clausius–Duhem inequality to obtain thermodynamic restrictions on constitutive relations goes back to Coleman and Noll's (1963) paper on thermoelastic materials with viscosity and has since been extended to other classes of materials by numerous authors. Of particular relevance are the papers by Wang and Bowen (1966) on quasi-elastic materials and Coleman and Gurtin (1967) on materials with internal state variables. These authors did not consider plasticity per se, but they did consider inelastic materials with properties similar to what we term instantaneous thermoelastic response. Mandel (1974) exploits a similar concept in his paper on "Thermodynamics and Plasticity," although his definition, which is a bit vague, is different than ours, and most details are omitted. Indeed, in papers on viscoplasticity, the derivation of the potential relations and the plastic dissipation inequality from the entropy inequality (4.53) is typically omitted, although the papers by Coleman and Gurtin (1967) and Green and Naghdi (1965) are often cited for details. As mentioned above, the former paper does not discuss plasticity per se. The analysis in the latter paper, while completely rigorous given the assumptions made there, is not applicable to the viscoplasticity theory considered here since it is based on the assumption that during plastic deformation the stress and temperature always lie on an evolving yield surface, inside of which the response is purely thermoelastic.[10]

In view of the above remarks and also because the property of instantaneous thermoelastic response involves assumptions that are weaker than those typically considered in papers on viscoplasticity,[11] our constitutive assumptions and their consequences are discussed at length in the next subsection, and detailed derivations of the thermodynamic restrictions are presented in Sections 4.5–4.7.

4.4. Instantaneous thermoelastic response

Several physical observations would seem to lead to a constitutive requirement of the sort that we term instantaneous thermoelastic response. First, infinitesimal ultrasonic waves have been observed to travel with their usual elastic wave speeds through regions that have previously been plastically deformed and that are being held in a state of incipient plastic deformation. Second, Bell and Stein (1962) confirmed earlier observations by Bell and others that the initial part of an incremental loading wave in a plastically prestressed bar also travels at the elastic speed. Similar results have been observed for reloading waves in normal plate impact tests (Herrmann, 1976). Third, so-called strain rate jump tests in a Hopkinson bar show an initially elastic transient as the flow stress switches continuously from one constant

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[10] For similar reasons the approach used by Casey (1998), which does not assume the existence of entropy as a primitive field but instead shows how an entropy function can be constructed, cannot be applied to the materials considered here.

[11] Under stronger assumptions of the type introduced in Section 5, the thermodynamic restrictions derived in Sections 4.5, 4.6, and 4.9 have been noted by numerous authors; for example, see Kratochvil (1971), Teodosiu and Sidoroff (1976), Davison et al. (1977), Anand (1985), and Cleja-Ţigoiu and Maugin (2000).

strain rate to another; see Duffy (1980) for a summary of a large amount of data, and also Brown et al. (1989).

The precise definition of instantaneous thermoelastic response is as follows. Consider a fixed material point **X**; for simplicity the argument **X** is suppressed below. For given initial values of q_n and $\mathbf{A}_k$ (recall that $\mathbf{F}_p = \mathbf{I}$ initially), we assume that $q_n(t)$, $\mathbf{A}_k(t)$, and $\mathbf{F}_p(t)$ depend only on the history of **F** and θ on the time interval $[0, t]$. In particular, q_n, $\mathbf{A}_k$, and $\mathbf{F}_p$ are independent of the current and past values of the temperature gradient, and hence so are their rates $\dot{q}_n$, $\dot{\mathbf{A}}_k$, and $\dot{\mathbf{F}}_p$. In addition, we assume that if the process $(\mathbf{F}(t), \theta(t))$, $t \geqslant 0$, is a piecewise continuous function of time, then q_n, $\mathbf{A}_k$, and $\mathbf{F}_p$ and their rates $\dot{q}_n$, $\dot{\mathbf{A}}_k$, and $\dot{\mathbf{F}}_p$ are continuous at any instant at which **F** and θ are continuous. In particular, the continuity of q_n, $\mathbf{A}_k$, $\mathbf{F}_p$, and their rates at the present time is unaffected by past jump discontinuities in **F** or θ, as would occur from the passage of shock waves through the point **X**, and is also unaffected by past or present jump discontinuities in $\dot{\mathbf{F}}$ and $\dot{\theta}$, as would occur from the passage of an acceleration wave. A material that satisfies the above restrictions and also the constitutive assumptions (4.34)–(4.36) for ψ, e, and **T** will be called a material with instantaneous thermoelastic response.

Observe that in the above definition we have made no assumptions about the continuity of q_n, $\mathbf{A}_k$, $\mathbf{F}_p$, and their rates at an instant when **F** and θ suffer jump discontinuities. Also note that the continuity assumptions on $\dot{\mathbf{F}}_p$ could not be satisfied if $\mathbf{F}_p$ were not invariant under superposed rigid motions. Suppose, for example, that $\mathbf{F}_p$ transformed as $\mathbf{F}_p^* = \mathbf{Q}\mathbf{F}_p$ under the superposed rigid motion $\mathbf{x}^* = \mathbf{Q}\mathbf{x}$, and consider the case where **F**, θ, and **Q** are continuous, but $\dot{\mathbf{Q}}$ is discontinuous. Then $\mathbf{F}^* = \mathbf{Q}\mathbf{F}$ and $\theta^* = \theta$ are continuous, and so are $\mathbf{F}_p$ and $\dot{\mathbf{F}}_p$ for a material with instantaneous thermoelastic response, but $(\mathbf{F}_p^*)^{\cdot} = \dot{\mathbf{Q}}\mathbf{F}_p + \mathbf{Q}\dot{\mathbf{F}}_p$ is discontinuous due to the discontinuity in $\dot{\mathbf{Q}}$. Thus the continuous process $(\mathbf{F}^*, \theta^*)$ would result in a plastic deformation gradient whose rate, $(\mathbf{F}_p^*)^{\cdot}$, is discontinuous, contrary to the definition of instantaneous thermoelastic response. Similarly, the continuity assumptions on $\dot{\mathbf{A}}_k$ would not be appropriate if these tensors were not invariant under superposed rigid motions; this point is discussed further Section 5.2.

As demonstrated at the end of this subsection, an example of a material with instantaneous thermoelastic response is one which satisfies the thermoelastic constitutive relations in Section 4.3 and evolution equations of the form

$$\dot{\mathbf{F}}_p = \mathcal{H}(\mathbf{F}_e, \theta, q_n, \mathbf{A}_k, \mathbf{F}_p), \tag{4.54}$$

$$\dot{q}_m = \xi_m(\mathbf{F}_e, \theta, q_n, \mathbf{A}_k, \mathbf{F}_p), \tag{4.55}$$

$$\dot{\mathbf{A}}_j = \mathcal{A}_j(\mathbf{F}_e, \theta, q_n, \mathbf{A}_k, \mathbf{F}_p), \tag{4.56}$$

for $m, n = 1, \ldots, N$ and $j, k = 1, \ldots, K$, provided the response functions $\mathcal{H}$, ξ_m, and $\mathcal{A}_j$ are "sufficiently nice." The materials considered in Section 5 are special cases of this class of materials. Observe that if the list of arguments on the right-hand side of

(4.54), (4.55), or (4.56) included $\dot{\theta}$ or some measure of the total rate of deformation, such as $\dot{\mathbf{F}}$, $\dot{\mathbf{E}}$, $\mathbf{L}$, $\mathbf{D}$, or $\tilde{\mathbf{D}}$, then the material would not have instantaneous thermoelastic response, since a discontinuity in these rates would result in a discontinuity in $\dot{q}_n$, $\dot{\mathbf{A}}_k$, or $\dot{\mathbf{F}}_\mathbf{p}$, even if $\mathbf{F}$ and θ are continuous. Also note that unlike the constitutive relations (4.34)–(4.36) for ψ, e, and $\mathbf{T}$, the plastic deformation gradient $\mathbf{F}_\mathbf{p}$ is included in the list of arguments for the evolution Eqs. (4.54)–(4.56). Elimination of $\mathbf{F}_\mathbf{p}$ from the right-hand side of (4.54) would be overly restrictive (see Section 5.1). On the other hand, some authors argue that $\mathbf{F}_\mathbf{p}$ (or more precisely, $\mathsf{F}_\mathbf{p}$) should be absent from the evolution equations for the internal variables, while others argue for its inclusion, often in the form of a plastic strain tensor. We will eventually adopt the former viewpoint, but in this section we allow an arbitrary dependence on $\mathbf{F}_\mathbf{p}$ in (4.54)–(4.56), if only to demonstrate that such dependence is not inconsistent with the property of instantaneous thermoelastic response and that it has no effect on the thermodynamic restrictions derived in Sections 4.5–4.7.

The qualitative properties that characterize a material with instantaneous thermoelastic response suffice for the derivation of the thermodynamic potentials and the plastic dissipation inequality from the entropy inequality (4.53). Indeed, the derivation of those results is not simplified in any way by restriction to the special class of materials satisfying (4.54)–(4.56). For this reason we postpone specialization to these evolution equations until Section 5.

Consider a material with instantaneous thermoelastic response undergoing a piecewise continuous process $(\mathbf{F}(t), \theta(t))$ for times $t \in [0, t_1 + \Delta t)$, with $\mathbf{F}$ and θ continuous at time t_1. Then q_n, $\mathbf{A}_k$, $\mathbf{F}_\mathbf{p}$ and $\dot{q}_n$, $\dot{\mathbf{A}}_k$, $\dot{\mathbf{F}}_\mathbf{p}$ are continuous at time t_1, by assumption. By (3.12), (3.13) and (4.16), $\rho_\mathbf{R}$, $\dot{\rho}_\mathbf{R}$, $\mathsf{F}_\mathbf{p}$, and $\dot{\mathsf{F}}_\mathbf{p}$ are also continuous at t_1. Since $\mathbf{F}_e = \mathbf{F}\mathbf{F}_\mathbf{p}^{-1}$, $\mathbf{F}_e$ is continuous at $t = t_1$, and hence so are $\mathbf{R}_e$, $\mathbf{V}_e$, $\mathbf{U}_e$, $\mathbf{C}_e$, and $\mathbf{E}_e$. From (4.45)–(4.48), ψ, e, η, and $\tilde{\mathbf{T}}$ are continuous at t_1, as are their partial derivatives with respect to $\mathbf{E}_e$, θ, q_n, and $\mathbf{A}_k$. By (4.36), the Cauchy stress tensor $\mathbf{T}$ is also continuous at t_1. From (4.14), (4.15), (4.18), (4.24), and (4.27), the plastic rates $\mathbf{L}_\mathbf{p}$, $\mathbf{D}_\mathbf{p}$, $\tilde{\mathbf{L}}_\mathbf{p}$, and $\tilde{\mathbf{D}}_\mathbf{p}$ and the plastic part $\mathcal{P}_\mathbf{p}$ of the stress power are continuous at time t_1. Observe that these properties hold regardless of any jump discontinuities in $\mathbf{F}$ and θ at times $t < t_1$ and regardless of possible discontinuities in $\dot{\mathbf{F}}$ and $\dot{\theta}$ at times $t \leqslant t_1$. If $\dot{\mathbf{F}}$ is also continuous at time t_1, then from $\mathbf{F}_e = \mathbf{F}\mathbf{F}_\mathbf{p}^{-1}$ and the continuity of $\mathbf{F}_\mathbf{p}$ and $\dot{\mathbf{F}}_\mathbf{p}$ at t_1 it follows that $\dot{\mathbf{F}}_e$ is continuous at t_1, and hence so are the elastic rates $\dot{\mathbf{R}}_e$, $\dot{\mathbf{V}}_e$, $\dot{\mathbf{U}}_e$, $\dot{\mathbf{C}}_e$, $\dot{\mathbf{E}}_e$, $\mathbf{L}_e$, $\mathbf{D}_e$, and $\tilde{\mathbf{L}}_e$, as well as the elastic and total stress power $\mathcal{P}_e$ and $\mathcal{P}$. Similarly, these variables are continuous from above (or below) at time t_1 if $\dot{\mathbf{F}}$ is continuous from above (or below) at t_1. If $\dot{\mathbf{F}}$ and $\dot{\theta}$ suffer jump discontinuities at time t_1, then $[\![\dot{\mathbf{F}}]\!] = [\![\dot{\mathbf{F}}_e]\!]\mathbf{F}_\mathbf{p}$, and by (4.51), $[\![\dot{\psi}]\!] = (\partial\psi/\partial\mathbf{E}_e) : [\![\dot{\mathbf{E}}_e]\!] + (\partial\psi/\partial\theta)[\![\dot{\theta}]\!]$, with analogous jump relations for $\dot{e}$, $\dot{\eta}$, and $\dot{\mathbf{T}}$. Thus the instantaneous response to jumps in the temperature rate and total deformation rate is thermoelastic.

It should be clear from the above discussion that once appropriate constitutive relations are provided for q_n, $\mathbf{A}_k$, and $\mathbf{F}_\mathbf{p}$, the theory is complete in the sense that the values of these variables, as well as the values of $\rho_\mathbf{R}$, $\mathbf{F}_e$, ψ, e, η, $\mathbf{T}$, and $\tilde{\mathbf{T}}$, are determined at any time t by the history of $\mathbf{F}$ and θ up to time t and the initial values of the internal variables. Note that we have placed no restrictions on the heat flux $\mathbf{q}$. Of course, a constitutive relation for the heat flux would have to be specified if the

energy equation is to be solved for the temperature distribution, but such problems are not considered in this paper.

In the remainder of this subsection, we demonstrate that a material satisfying the thermoelastic constitutive relations in Section 4.3 and the evolution Eqs. (4.54)–(4.56) has the property of instantaneous thermoelastic response, provided the constitutive functions $\mathcal{H}$, ξ_m, and $\mathcal{A}_j$ in these evolution equations are Lipschitz continuous on any compact subset of their domain. Typically they would also be piecewise smooth.[12] Since $\mathbf{F}_e = \mathbf{F}\mathbf{F}_p^{-1}$, Eqs. (4.54)–(4.56) are equivalent to the evolution equations

$$\dot{\mathbf{F}}_p = \mathcal{H}^{\#}(\mathbf{F}, \theta, q_n, \mathbf{A}_k, \mathbf{F}_p), \tag{4.57}$$

$$\dot{q}_m = \xi_m^{\#}(\mathbf{F}, \theta, q_n, \mathbf{A}_k, \mathbf{F}_p), \tag{4.58}$$

$$\dot{\mathbf{A}}_j = \mathcal{A}_j^{\#}(\mathbf{F}, \theta, q_n, \mathbf{A}_k, \mathbf{F}_p), \tag{4.59}$$

where $\mathcal{H}^{\#}(\mathbf{F}, \theta, q_n, \mathbf{A}_k, \mathbf{F}_p) = \mathcal{H}\left(\mathbf{F}\mathbf{F}_p^{-1}, \theta, q_n, \mathbf{A}_k, \mathbf{F}_p\right)$ for example, and the functions $\mathcal{H}^{\#}$, $\xi_m^{\#}$, and $\mathcal{A}_j^{\#}$ are also Lipschitz continuous on any compact subset of their domain.

For a fixed material point and a given piecewise continuous process $(\mathbf{F}(t), \theta(t))$ for $t \geqslant 0$, the equivalent system (4.57)–(4.59) is a first-order system of ordinary differential equations for the dependent variables $\mathbf{F}_p$, q_n, and $\mathbf{A}_k$, say

$$\dot{\mathbf{F}}_p(t) = \mathcal{H}^{\natural}(t, q_n(t), \mathbf{A}_k(t), \mathbf{F}_p(t)), \tag{4.60}$$

$$\dot{q}_m(t) = \xi_m^{\natural}(t, q_n(t), \mathbf{A}_k(t), \mathbf{F}_p(t)), \tag{4.61}$$

$$\dot{\mathbf{A}}_j(t) = \mathcal{A}_j^{\natural}(t, q_n(t), \mathbf{A}_k(t), \mathbf{F}_p(t)). \tag{4.62}$$

The functions $\mathcal{H}^{\natural}$, $\xi_m^{\natural}$, and $\mathcal{A}_j^{\natural}$ depend on the functions $\mathcal{H}^{\#}$, $\xi_m^{\#}$, and $\mathcal{A}_j^{\#}$ and also on the particular process $(\mathbf{F}, \theta)$, for example,

$$\mathcal{H}^{\natural}(t, q_n, \mathbf{A}_k, \mathbf{F}_p) = \mathcal{H}^{\#}(\mathbf{F}(t), \theta(t), q_n, \mathbf{A}_k, \mathbf{F}_p). \tag{4.63}$$

$\mathcal{H}^{\#}$, $\xi_m^{\natural}$, and $\mathcal{A}_j^{\natural}$ are at least piecewise continuous with possible jump discontinuities across the planes $t = \bar{t}$ in $(t, q_n, \mathbf{A}_k, \mathbf{F}_p)$-space, where $\bar{t}$ is any one of the discrete

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[12] Requiring these functions to be continuously differentiable at all points would be too restrictive an assumption in general. For materials with an explicit yield surface (see Section 5.3), $\mathcal{H}$ is $\mathbf{0}$ inside the yield surface and would typically be smooth outside the yield surface, but its partial derivatives could suffer jump discontinuities at the yield surface.

instants at which $\mathbf{F}$ or θ suffer jump discontinuities. Furthermore, on any compact subset of their domain in $(t, q_n, \mathbf{A}_k, \mathbf{F}_p)$-space, the functions $\mathcal{H}^\natural$, $\xi_m^\natural$, and $\mathcal{A}^{j\natural}$ satisfy a Lipschitz condition in the dependent variables q_n, $\mathbf{A}_k$, and $\mathbf{F}_p$. For example,

$$
\begin{aligned}
&\left\| \mathcal{H}^\natural\big(t, q_n, \mathbf{A}_k, \mathbf{F}_p\big) - \mathcal{H}^\natural\big(t, \bar{q}_n, \bar{\mathbf{A}}_k, \bar{\mathbf{F}}_p\big) \right\| \\
&= \left\| \mathcal{H}^\#\big(\mathbf{F}(t), \theta(t), q_n, \mathbf{A}_k, \mathbf{F}_p\big) - \mathcal{H}^\#\big(\mathbf{F}(t), \theta(t), \bar{q}_n, \bar{\mathbf{A}}_k, \bar{\mathbf{F}}_p\big) \right\| \\
&\leqslant \sum_{n=1}^{N} M_n \big| q_n - \bar{q}_n \big| + \sum_{k=1}^{K} \hat{M}_k \big\| \mathbf{A}_k - \bar{\mathbf{A}}_k \big\| + M \big\| \mathbf{F}_p - \bar{\mathbf{F}}_p \big\|
\end{aligned}
$$

for some constants M_n, $\hat{M}_k$, and M (independent of t), by the Lipschitz continuity of the function $\mathcal{H}^\#$.

For given initial values of the internal variables (recall that $\mathbf{F}_p = \mathbf{I}$ initially), these properties suffice for the existence of a unique solution q_n, $\mathbf{A}_k$, $\mathbf{F}_p$ of (4.60)–(4.62) that is continuous and piecewise smooth.[13] The only possible discontinuities in $\dot{\mathbf{F}}_p$, $\dot{q}_n$, or $\dot{\mathbf{A}}_k$ are the jump discontinuities that can occur at an instant $\bar{t}$ at which $\mathbf{F}$ or θ suffer a jump discontinuity, in which case (4.54)–(4.56) hold for the one-sided derivatives of $\mathbf{F}_p$, q_n, and $\mathbf{A}_j$. Clearly, the conditions of instantaneous thermoelastic response are satisfied. In addition, $\mathbf{F}_p$, q_n, and $\mathbf{A}_k$ are continuous across a shock, whereas their rates suffer at most jump discontinuities across a shock. Note that these shock properties are derived results for this particular class of materials — they were not assumed in the definition of instantaneous thermoelastic response and do not follow from it without additional assumptions.

4.5. The thermoelastic potentials

Henceforth, we assume the material has instantaneous thermoelastic response. For a fixed material point $\mathbf{X}$, consider a piecewise continuous process $(\mathbf{F}(t), \theta(t))$ for $t \in [0, t_1]$, with $\mathbf{F}$ and θ continuous (from below) at time t_1. Choose any tensor $\mathbf{G}$ and scalar Θ, and extend the process $(\mathbf{F}(t), \theta(t))$ to the time interval $(t_1, t_1 + \Delta t)$ by defining

$$
\mathbf{F}(t) = \mathbf{F}(t_1) + (t - t_1)\mathbf{G}, \quad \theta(t) = \theta(t_1) + (t - t_1)\Theta, \quad t \in (t_1, t_1 + \Delta t), \tag{4.64}
$$

where the time increment $\Delta t > 0$ is sufficiently small so that $\det\mathbf{F}(t)$ and $\theta(t)$ are positive. Then the extended process is continuous at each time $t \in [t_1, t_1 + \Delta t)$ and is piecewise continuous on the time interval $[0, t_1 + \Delta t)$. Hence, by the property of

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[13] The above assumptions suffice for the existence and uniqueness proof by the method of successive approximation (Ince, 1956, Section 3.21), which is usually stated with the stronger assumption of continuity of $\mathcal{H}^\natural$, $\xi_m^\natural$, and $\mathcal{A}_j^\natural$ (along with the Lipschitz condition in the dependent variables).

instantaneous thermoelastic response, the terms in parentheses in the entropy inequality (4.53) are continuous at each time $t \in [t_1, t_1 + \Delta t)$ and, by (4.64), $\dot{\mathbf{F}}$ and $\dot{\theta}$ are continuous on the open interval $(t_1, t_1 + \Delta t)$ with limits from above at t_1, namely $\dot{\mathbf{F}}(t_1^+) = \mathbf{G}$ and $\dot{\theta}(t_1^+) = \Theta$. Since these properties are independent of the history of the temperature gradient at $\mathbf{X}$, they certainly hold if the process is homothermal at $\mathbf{X}$, that is, if $\nabla\theta(\mathbf{X}, t) = 0$ for all $t \in [0, t_1 + \Delta t)$.[14] For a homothermal process the term involving the heat flux drops out of the entropy inequality (4.53).[15] On evaluating this inequality at $t \in (t_1, t_1 + \Delta t)$ and taking the limit as $t \to t_1$ from above, we obtain

$$\left(\frac{\partial\psi}{\partial\mathbf{E}_e} - \frac{\tilde{\mathbf{T}}}{\rho_R}\right)(t_1) : \dot{\mathbf{E}}_e(t_1^+) + \left(\frac{\partial\psi}{\partial\theta} + \eta\right)(t_1)\Theta \leqslant \left(\mathcal{P}_p - \sum_{n=1}^{N}\frac{\partial\psi}{\partial q_n}\dot{q}_n - \sum_{k=1}^{K}\frac{\partial\psi}{\partial\mathbf{A}_k} : \dot{\mathbf{A}}_k\right)(t_1).$$

$$(4.65)$$

Since each of the terms in (4.65) is independent of the temperature gradient, this inequality holds regardless of whether or not the process is homothermal.

From (4.5), (4.25), and (4.28), we have, in general,

$$\text{sym}\left(\mathbf{F}^T\dot{\mathbf{F}}\right) = \dot{\mathbf{E}} = \mathbf{F}_p^T\left(\dot{\mathbf{E}}_e + \tilde{\mathbf{D}}_p\right)\mathbf{F}_p, \qquad (4.66)$$

so for the linear extension (4.64), $\mathbf{G}$ and $\dot{\mathbf{E}}_e(t_1^+)$ are related by

$$\text{sym}\left(\mathbf{F}(t_1)^T\mathbf{G}\right) = \mathbf{F}_p(t_1)^T\left(\dot{\mathbf{E}}_e(t_1) + \tilde{\mathbf{D}}_p(t_1)\right)\mathbf{F}_p(t_1). \qquad (4.67)$$

For any appropriate values of $\mathbf{F}(t_1)$, $\mathbf{F}_p(t_1)$, and $\tilde{\mathbf{D}}_p(t_1)$, we can assign the value of any symmetric tensor to $\dot{\mathbf{E}}_e(t_1^+)$ by choosing $\mathbf{G}$ so that (4.67) is satisfied, for example, by setting

$$\mathbf{G} = \mathbf{F}(t_1)^{-T}\mathbf{F}_p(t_1)^T\left(\dot{\mathbf{E}}_e(t_1^+) + \tilde{\mathbf{D}}_p(t_1)\right)\mathbf{F}_p(t_1). \qquad (4.68)$$

Thus, (4.65) holds for any symmetric tensor $\dot{\mathbf{E}}_e(t_1^+)$ and scalar Θ, whereas the other terms are independent of $\dot{\mathbf{E}}_e(t_1^+)$ and Θ. Therefore, we must have $\left(\partial\psi/\partial\mathbf{E}_e - \tilde{\mathbf{T}}/\rho_R\right)(t_1) = \mathbf{0}$ and $(\partial\psi/\partial\theta + \eta)(t_1) = 0$. Since the time t_1 and the piecewise continuous process $(\mathbf{F}(t), \theta(t))$, $t \in [0, t_1]$, are arbitrary, these restrictions must

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[14] We assume that the process described here is dynamically possible in the sense that the body force $\mathbf{b}$ and energy supply rate r needed to satisfy balance of linear momentum (4.1) and balance of energy (4.2) can be applied, at least in principle.

[15] Actually, this conclusion requires the assumption that the heat flux $\mathbf{q}$ be bounded for all homothermal processes of the type considered here. Since it is reasonable to expect that the heat flux should be zero for homothermal processes, this condition is extremely weak.

hold for all possible values that $\mathbf{E}_e$, θ, q_n, and $\mathbf{A}_k$ take on in any such processes. Thus $\tilde{\mathbf{T}}$ and η must satisfy

$$\tilde{\mathbf{T}} = \rho_R \frac{\partial \psi}{\partial \mathbf{E}_e}(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k), \quad \eta = -\frac{\partial \psi}{\partial \theta}(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k). \tag{4.69}$$

The specific heat (per unit mass) at constant elastic strain is defined by

$$C_{\mathbf{E}_e} = \frac{\partial e}{\partial \theta} = \theta \frac{\partial \eta}{\partial \theta} = -\theta \frac{\partial^2 \psi}{\partial \theta^2}, \tag{4.70}$$

where the second and third relations follow from $e = \psi + \theta\eta$ and the potential relation $(4.69)_2$. We assume that $C_{\mathbf{E}_e} > 0$. This is equivalent to assuming $\partial\eta/\partial\theta > 0$, which in turn is equivalent to the condition that entropy is a strictly increasing function of temperature for fixed $\mathbf{E}_e$, q_n, and $\mathbf{A}_k$. Thus η is invertible in θ, and we have

$$\theta = \bar{\theta}(\mathbf{E}_e, \eta, q_n, \mathbf{A}_k), \quad e = \bar{e}(\mathbf{E}_e, \eta, q_n, \mathbf{A}_k). \tag{4.71}$$

Then, from $e = \psi + \theta\eta$, (4.69), and the chain rule, it follows that[16]

$$\tilde{\mathbf{T}} = \rho_R \frac{\partial e}{\partial \mathbf{E}_e}(\mathbf{E}_e, \eta, q_n, \mathbf{A}_k), \quad \theta = \frac{\partial e}{\partial \eta}(\mathbf{E}_e, \eta, q_n, \mathbf{A}_k). \tag{4.72}$$

In particular, $\partial e/\partial \eta > 0$, which is equivalent to the condition that internal energy is a strictly increasing function of entropy for fixed $\mathbf{E}_e$, q_n, and $\mathbf{A}_k$. Thus e is invertible in η, and we have

$$\eta = \check{\eta}(\mathbf{E}_e, e, q_n, \mathbf{A}_k), \quad \theta = \check{\theta}(\mathbf{E}_e, e, q_n, \mathbf{A}_k). \tag{4.73}$$

Then, (4.72) implies

$$\tilde{\mathbf{T}} = -\rho_R \theta \frac{\partial \eta}{\partial \mathbf{E}_e}(\mathbf{E}_e, e, q_n, \mathbf{A}_k), \quad \frac{1}{\theta} = \frac{\partial \eta}{\partial e}(\mathbf{E}_e, e, q_n, \mathbf{A}_k). \tag{4.74}$$

The relations (4.69), (4.72), and (4.74) generalize the potentials of classical thermoelasticity by measuring the elastic strain relative to a time dependent, local

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[16] An alternate approach is to take entropy as the independent thermodynamic variable from the start. Then the entropy inequality becomes (4.53) with ψ replaced by e on the right-hand side, and the left-hand side replaced by $\left(\partial e/\partial \mathbf{E}_e - \tilde{\mathbf{T}}/\rho_R\right) : \dot{\mathbf{E}}_e + (\partial e/\partial \eta - \theta)\dot{\eta}$. If θ is replaced by η in the definition of instantaneous thermoelastic response, then arguments similar to those above yield (4.72) directly, in which case (4.69) follows from the assumption $C_{\mathbf{E}_e} > 0$.

reference configuration and by including dependence on the evolving internal structure.[17]

4.6. The plastic dissipation inequality

From the potential relations and the chain rule, we find that the partial derivatives of ψ, e, and η with respect to the internal variables are related as follows:

$$-\theta\left(\frac{\partial \eta}{\partial q_n}\right)_e = \left(\frac{\partial e}{\partial q_n}\right)_\eta = \left(\frac{\partial \psi}{\partial q_n}\right)_\theta \equiv \frac{Q_n}{\rho_R},$$

(4.75)

$$-\theta\left(\frac{\partial \eta}{\partial \mathbf{A}_k}\right)_e = \left(\frac{\partial e}{\partial \mathbf{A}_k}\right)_\eta = \left(\frac{\partial \psi}{\partial \mathbf{A}_k}\right)_\theta \equiv \frac{\Lambda_k}{\rho_R},$$

(4.76)

where the e, η, and θ subscripts indicate which thermodynamic variable is being held constant. From the expression (4.52) for $\dot{\mathcal{W}}_s$, the potential relations (4.69), and the relations (4.75) and (4.76), we see that

$$\dot{\mathcal{W}}_s = \sum_{n=1}^{N}\left(\frac{\partial \psi}{\partial q_n}\right)_\theta \dot{q}_n + \sum_{k=1}^{K}\left(\frac{\partial \psi}{\partial \mathbf{A}_k}\right)_\theta : \dot{\mathbf{A}}_k$$

$$= \sum_{n=1}^{N}\left(\frac{\partial e}{\partial q_n}\right)_\eta \dot{q}_n + \sum_{k=1}^{K}\left(\frac{\partial e}{\partial \mathbf{A}_k}\right)_\eta : \dot{\mathbf{A}}_k = \frac{1}{\rho_R}\sum_{n=1}^{N}Q_n\dot{q}_n + \frac{1}{\rho_R}\sum_{k=1}^{K}\Lambda_k : \dot{\mathbf{A}}_k.$$

(4.77)

According to the concept of field averages and fluctuations, the elastic stress power $\mathcal{P}_e$ is the rate of change of elastic energy (per unit mass) due to mechanical power of the average stresses or "long range forces," while from $(4.77)_{1,2}$ it is seen that $\dot{\mathcal{W}}_s$ represents the rate of elastic energy storage (per unit mass) due to the fluctuations around dislocations or other "short range forces." Since $\dot{\mathcal{W}}_s$, although arising from elastic changes, only exists when the internal structure is changing, it may be associated with the rate of change of the stored energy of cold work (and perhaps also with the localized elastic energy stored in the lattice around point defects). Q_n and Λ_k (or the derivatives of the potentials with respect to the internal variables) may be regarded as conjugate internal forces that do work against changes in internal variables.

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[17] Eqs. (4.30) and (4.31) and the potential relations for $\tilde{\mathbf{T}}$ yield expressions for the elastic and plastic stress power in terms of the potential functions; e.g. $\mathcal{P}_p = \partial\psi/\partial\mathbf{E}_e : \tilde{\mathbf{D}}_p$. Some authors (Rajagopal and Srinivasa, 1998b) prefer to work with the total strain tensor $\mathbf{E}$ and the second Piola–Kirchhoff stress tensor $\tilde{\mathbf{T}}_0$ relative to the initial configuration. From the relations in Sections 3.1–3.2, we find that $\tilde{\mathbf{T}} = \left(\frac{\rho_R}{\rho_0}\right)\mathbf{F}_p\tilde{\mathbf{T}}_0\mathbf{F}_p^T$ and $\mathbf{C}_e = \mathbf{F}_p^{-T}\mathbf{C}\mathbf{F}_p^{-1}$. Then the expression (4.31) for the plastic stress power yields $\rho_0\mathcal{P}_p = \mathbf{C}\tilde{\mathbf{T}}_0 : \mathbf{F}_p^{-1}\dot{\mathbf{F}}_p$. From the above relation for $\mathbf{C}_e$ and $\mathbf{C} = 2\mathbf{E} + \mathbf{I}$, we obtain an expression for $\mathbf{E}_e$ in terms of $\mathbf{E}$ and $\mathbf{F}_p$: $2\mathbf{E}_e = \mathbf{F}_p^{-T}(2\mathbf{E} + \mathbf{I})\mathbf{F}_p^{-1} - \mathbf{I}$. Then $\psi = \psi^\#(\mathbf{E}, \mathbf{F}_p, \theta, q_n, \mathbf{A}_k)$, for example, and $(4.69)_1$ implies the potential relations $\tilde{\mathbf{T}}_0 = \rho_0\partial\psi^\#/\partial\mathbf{E}$ and $\tilde{\mathbf{T}}_0 = -\rho_0\mathbf{C}^{-1}\mathbf{F}_p^T\partial\psi^\#/\partial\mathbf{F}_p$. The latter relation and the above relation for $\rho_0\mathcal{P}_p$ yield $\mathcal{P}_p = -\frac{\partial\psi^\#}{\partial\mathbf{F}_p} : \dot{\mathbf{F}}_p$.

Since the left-hand side of the inequality (4.65) is zero, by (4.77)$_1$ and (4.19) this inequality reduces to

$$\delta = \mathcal{P}_\mathrm{p} - \dot{\mathcal{W}}_\mathrm{s} \geqslant 0, \tag{4.78}$$

that is, the internal dissipation δ is nonnegative for all processes. This may be referred to as the plastic dissipation inequality. When written as $\dot{\mathcal{W}}_\mathrm{s} \leqslant \mathcal{P}_\mathrm{p}$, the plastic dissipation inequality states that the rate of change of the stored energy of cold work cannot exceed the rate of plastic work. Recall that expressions for the plastic stress power (per unit mass), $\mathcal{P}_\mathrm{p}$, are given by (4.18), (4.30) and (4.31); alternate expressions are derived in Section 4.8 below.

The plastic dissipation inequality (4.78) holds regardless of the value of the temperature gradient. When $\nabla\theta \neq 0$, the entropy inequality (4.21) must also be satisfied. For purely thermoelastic processes, this reduces to the heat conduction inequality

$$\mathbf{q} \cdot \nabla\theta \leqslant 0. \tag{4.79}$$

However, for inelastic processes (4.79) does not follow from the entropy and plastic dissipation inequalities given the constitutive assumptions made up to this point. Nevertheless, the heat conduction inequality (4.79) may be adopted as a separate constitutive assumption since it is not inconsistent with the entropy inequality or the plastic dissipation inequality.

For a piecewise continuous process $(\mathbf{F}(t), \theta(t))$, the property of instantaneous thermoelastic response guarantees that the plastic stress power $\mathcal{P}_\mathrm{p}$, the rate of storage of cold work $\dot{\mathcal{W}}_\mathrm{s}$, and the internal dissipation $\mathcal{P}_\mathrm{p} - \dot{\mathcal{W}}_\mathrm{s}$ are continuous at an instant when $\mathbf{F}$ and θ are continuous. Although the internal dissipation depends on the rate of plastic deformation, it cannot depend on the total rate of deformation (e.g. $\dot{\mathbf{F}}$ or $\dot{\mathbf{E}}$) or on the rate of elastic deformation (e.g. $\dot{\mathbf{E}}_\mathrm{e}$), since these quantities need not be continuous when $\mathbf{F}$ and θ are continuous.

4.7. Balance of energy and thermodynamic restrictions on ρ_R

Recall that conservation of mass from the initial to the intermediate configuration is expressed by $\rho_\mathrm{R} = \rho_0/\det\mathbf{F}_\mathrm{p}$ [see (3.12)$_1$], or in rate form by (4.16). As noted in Section 4.4, for a material with instantaneous thermoelastic response these relations and the continuity of $\mathbf{F}_\mathrm{p}$ and $\dot{\mathbf{F}}_\mathrm{p}$ at an instant when $\mathbf{F}$ and θ are continuous imply the continuity of ρ_R and $\dot{\rho}_\mathrm{R}$ at that instant. Similarly, since $\mathbf{F}_\mathrm{p}(t)$ depends only on the initial values of the internal variables and the history of $\mathbf{F}$ and θ up to time t, so does $\rho_\mathrm{R}(t)$. No additional properties of ρ_R have been assumed or derived up to this point. In particular, we have not assumed that ρ_R is an internal variable or even that it can be expressed as a function of the internal variables.

We will now show that the property of instantaneous thermoelastic response implies that the density ρ_R in the intermediate configuration can be at most a function of the current values of the internal state variables, say

$$\rho_R = \hat{\rho}_R(q_n, A_k).$$ (4.80)

To see this, first recall that the constitutive assumption for the Cauchy stress, (4.36), yields the reduced constitutive relation (4.48) for $\tilde{T}$ when invariance properties are taken into account and, since $\tilde{T}$ must also satisfy the potential relation (4.69)$_1$, we have

$$\tilde{T} = \tilde{\tau}(E_e, \theta, q_n, A_k) = \rho_R \frac{\partial \psi}{\partial E_e}(E_e, \theta, q_n, A_k).$$ (4.81)

It follows that ρ_R can be at most a function of E_e, θ, q_n, and A_k. Therefore,

$$\dot{\rho}_R = \frac{\partial \rho_R}{\partial E_e} : \dot{E}_e + \frac{\partial \rho_R}{\partial \theta} \dot{\theta} + \sum_{n=1}^{N} \frac{\partial \rho_R}{\partial q_n} \dot{q}_n + \sum_{k=1}^{K} \frac{\partial \rho_R}{\partial A_k} : \dot{A}_k.$$

Now consider a process for which F and θ are continuous, with $\dot{F}$ and $\dot{\theta}$ possibly suffering jump discontinuities at time t_1. By assumption, $\dot{q}_n$ and $\dot{A}_k$ are continuous for such a process, and as noted above, $\dot{\rho}_R$ is also continuous, so that at time t_1,

$$0 = [\![\dot{\rho}_R]\!] = \frac{\partial \rho_R}{\partial E_e} : [\![\dot{E}_e]\!] + \frac{\partial \rho_R}{\partial \theta} [\![\dot{\theta}]\!].$$

Proceeding as in the derivation of the potential relations, we see that for given values of E_e, θ, q_n, and A_k, the jumps $[\![\dot{E}_e]\!](t_1)$ and $[\![\dot{\theta}]\!](t_1)$ can be varied arbitrarily by varying G and Θ in the continuous linear extension (4.64), and hence the above equation can hold only if $\partial \rho_R / \partial E_e = 0$ and $\partial \rho_R / \partial \theta = 0$, which proves the claim.

Observe that (4.80) is consistent with the ideas expressed at the end of Section 2. Also note that the general relation (4.80) includes the special case where ρ_R is itself an internal variable, say $\rho_R = q_1$. This special case is discussed further at the end of the next two subsections.

Since ρ_R is independent of θ, the potential relations (4.69) and the definitions (4.75)$_3$ and (4.76)$_3$ of Q_n and Λ_k yield the relations

$$\left(\frac{\partial \eta}{\partial E_e}\right)_\theta = -\frac{1}{\rho_R} \frac{\partial \tilde{T}}{\partial \theta}, \quad \left(\frac{\partial \eta}{\partial q_n}\right)_\theta = -\frac{1}{\rho_R} \frac{\partial Q_n}{\partial \theta}, \quad \left(\frac{\partial \eta}{\partial A_k}\right)_\theta = -\frac{1}{\rho_R} \frac{\partial \Lambda_k}{\partial \theta}.$$ (4.82)

On using $\delta = \mathcal{P}_p - \dot{\mathcal{W}}_s$ and the last expression for $\dot{\mathcal{W}}_s$ in (4.77), the energy balance Eq. (4.11) may be written as

$$\theta \dot{\eta} = -\frac{\text{div } q}{\rho} + r + \mathcal{P}_p - \frac{1}{\rho_R} \sum_{n=1}^{N} Q_n \dot{q}_n - \frac{1}{\rho_R} \sum_{k=1}^{K} \Lambda_k : \dot{A}_k.$$ (4.83)

On expanding the $\dot{\eta}$ term in (4.83) and using (4.82) and (4.70), we find that balance of energy may also be expressed as

$$C_{E_e} \dot{\theta} = -\frac{\text{div } \mathbf{q}}{\rho} + r + \frac{\theta}{\rho_R} \frac{\partial \tilde{\mathbf{T}}}{\partial \theta} : \dot{\mathbf{E}}_e + \Delta, \tag{4.84}$$

where

$$\Delta = \mathcal{P}_p - \frac{1}{\rho_R} \sum_{n=1}^{N} \left(Q_n - \theta \frac{\partial Q_n}{\partial \theta} \right) \dot{q}_n - \frac{1}{\rho_R} \sum_{k=1}^{K} \left(\Lambda_k - \theta \frac{\partial \Lambda_k}{\partial \theta} \right) : \dot{\mathbf{A}}_k$$

$$= \delta + \frac{\theta}{\rho_R} \sum_{n=1}^{N} \frac{\partial Q_n}{\partial \theta} \dot{q}_n + \frac{\theta}{\rho_R} \sum_{k=1}^{K} \frac{\partial \Lambda_k}{\partial \theta} : \dot{\mathbf{A}}_k \tag{4.85}$$

may be referred to as the inelastic heating (per unit mass). When $\Delta = 0$, the balance of energy reduces exactly to the thermoelastic case, but now the intermediate configuration for each material point serves as the local reference configuration.

4.8. Decomposition of the plastic stress power

In view of the fact that $\rho_R = \hat{\rho}_R(q_n, \mathbf{A}_k)$, it is useful to decompose the plastic stress power and the various measures of rate of plastic deformation into separate contributions from plastic volume change and dislocation slip. For the tensors $\mathbf{D}_p$ and $\dot{\mathbf{F}}_p \mathbf{F}_p^{-1}$, such a decomposition involves their spherical and deviatoric parts. From (4.15), (4.16), and (3.13), we have

$$\mathbf{D}_p = \mathsf{D}_p - \frac{1}{3} \frac{\dot{\rho}_R}{\rho_R} \mathbf{I}, \quad \text{tr} \mathbf{D}_p = 0, \tag{4.86}$$

which defines the deviatoric measure $\mathbf{D}_p$ of plastic shearing, and

$$\dot{\mathbf{F}}_p \mathbf{F}_p^{-1} = \dot{\mathbf{F}}_p \mathbf{F}_p^{-1} - \frac{1}{3} \frac{\dot{\rho}_R}{\rho_R} \mathbf{I}, \quad \text{tr} \left(\dot{\mathbf{F}}_p \mathbf{F}_p^{-1} \right) = 0, \tag{4.87}$$

From these relations and (4.14) and (4.15), we obtain an alternate expression for $\mathbf{D}_p$ that is analogous to the definition (4.15) of $\mathbf{D}_p$,

$$\mathsf{D}_p = \text{sym} \left[\mathbf{F}_e \left(\dot{\mathbf{F}}_p \mathbf{F}_p^{-1} \right) \mathbf{F}_e^{-1} \right]. \tag{4.88}$$

By (4.16), $-\dot{\rho}_R / \rho_R = \text{tr} \mathbf{D}_p = \text{tr} \left(\dot{\mathbf{F}}_p \mathbf{F}_p^{-1} \right)$. On the other hand, since $\rho_R = \hat{\rho}_R(q_n, \mathbf{A}_k)$, we have

$$\frac{\dot{\rho}_R}{\rho_R} = \frac{1}{\rho_R} \sum_{n=1}^{N} \frac{\partial \rho_R}{\partial q_n} \dot{q}_n + \frac{1}{\rho_R} \sum_{k=1}^{K} \frac{\partial \rho_R}{\partial \mathbf{A}_k} : \dot{\mathbf{A}}_k. \tag{4.89}$$

Note that in the special case where ρ_R is itself an internal variable, say $\rho_R = q_1$, Eq. (4.89) simply reduces to $\dot{\rho}_R/\rho_R = \dot{q}_1/q_1$.

The Cauchy stress may be decomposed into a deviatoric part, $\mathbf{S}$, which is a measure of shear stress, and a spherical part, $-p\mathbf{I}$, where p is the pressure,

$$\mathbf{T} = \mathbf{S} - p\mathbf{I}, \quad \mathrm{tr}\mathbf{S} = 0, \quad p = -\frac{1}{3}\mathrm{tr}\mathbf{T}. \tag{4.90}$$

Then by (4.18), (4.86) and (4.90), the plastic stress power (per unit mass) may be decomposed into a contribution $\mathcal{P}_p^s$ from plastic shearing due to dislocation slip and a contribution $\mathcal{P}_p^v$ from the rate of plastic volume change:

$$\mathcal{P}_p = \mathcal{P}_p^s + \mathcal{P}_p^v, \tag{4.91}$$

where

$$\mathcal{P}_p^s = \frac{\mathbf{T}:\mathbf{D}_p}{\rho} = \frac{\mathbf{S}:\mathbf{D}_p}{\rho}, \quad \mathcal{P}_p^v = \frac{p\,\dot{\rho}_R}{\rho\,\rho_R}. \tag{4.92}$$

By analogy with the definition (3.14) of $\tilde{\mathbf{T}}$, we introduce the stress tensor

$$\tilde{\mathbf{S}} = (\det\mathbf{F}_e)\mathbf{F}_e^{-1}\mathbf{S}\mathbf{F}_e^{-T} = \frac{\rho_R}{\rho}\mathbf{F}_e^{-1}\mathbf{S}\mathbf{F}_e^{-T} \approx \left(\frac{\rho_R}{\rho}\right)^{1/3}\mathbf{R}_e^{T}\mathbf{S}\mathbf{R}_e. \tag{4.93}$$

Here and below, the approximations are for small elastic shear strains. By analogy with the definition (4.27) of $\tilde{\mathbf{D}}_P$, we introduce the plastic shearing tensor

$$\tilde{\mathbf{D}}_p = \mathbf{F}_e^{T}\mathbf{D}_p\mathbf{F}_e \approx \left(\frac{\rho_R}{\rho}\right)^{2/3}\mathbf{R}_e^{T}\mathbf{D}_p\mathbf{R}_e. \tag{4.94}$$

By (4.94)$_1$ and (4.88), we also have

$$\tilde{\mathbf{D}}_p = \mathrm{sym}\left(\mathbf{C}_e\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}\right). \tag{4.95}$$

For plastically incompressible materials (or more generally, whenever $\dot{\rho}_R = 0$), $\mathbf{D}_p$ coincides with $\mathbf{D}_p$ and $\tilde{\mathbf{D}}_p$ coincides with $\tilde{\mathbf{D}}_p$. Note that unlike $\mathbf{S}$ and $\mathbf{D}_p$, the tensors $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{D}}_p$ are invariant under superposed rigid motions and generally not deviatoric. Instead, they satisfy the constraints

$$\mathrm{tr}\left(\tilde{\mathbf{S}}\mathbf{C}_e\right) = 0, \quad \mathrm{tr}\left(\tilde{\mathbf{D}}_p\mathbf{C}_e^{-1}\right) = 0. \tag{4.96}$$

On the other hand, it is clear from (4.94)$_2$ and (4.93)$_3$ that $\tilde{\mathbf{D}}_p$ and $\tilde{\mathbf{S}}$ are approximately deviatoric for small elastic shear strains. Also note that the analogs of (4.90)$_{1,3}$ and (4.86)$_1$ are

$$\tilde{\mathbf{T}} = \tilde{\mathbf{S}} - p\frac{\rho_R}{\rho}\mathbf{C}_e^{-1} \approx \tilde{\mathbf{S}} - \tilde{p}\mathbf{I}, \quad \tilde{p} \equiv -\frac{1}{3}\mathrm{tr}\tilde{\mathbf{T}}, \tag{4.97}$$

and

$$\tilde{\mathbf{D}}_p = \check{\mathbf{D}}_p - \frac{1}{3}\frac{\dot{\rho}_R}{\rho_R}\mathbf{C}_e \approx \check{\mathbf{D}}_p - \frac{1}{3}\left(\frac{\rho_R}{\rho}\right)^{2/3}\frac{\dot{\rho}_R}{\rho_R}\mathbf{I}. \tag{4.98}$$

From the above relations we obtain expressions for $\mathcal{P}_p^s$ in terms of quantities in the intermediate configuration,

$$\rho_R \mathcal{P}_p^s = \tilde{\mathbf{T}} : \tilde{\mathbf{D}}_p = \tilde{\mathbf{S}} : \check{\mathbf{D}}_p. \tag{4.99}$$

These relations may also be written as

$$\rho_R \mathcal{P}_p^s = \mathbf{C}_e \tilde{\mathbf{T}} : \dot{\mathbf{F}}_p \mathbf{F}_p^{-1} = \mathbf{C}_e \tilde{\mathbf{S}} : \dot{\mathbf{F}}_p \mathbf{F}_p^{-1}. \tag{4.100}$$

Alternate expressions for the term p/ρ in (4.92) and in (4.103) below follow from the identities

$$-3\frac{p}{\rho} = \frac{\mathrm{tr}\mathbf{T}}{\rho} = \frac{\tilde{\mathbf{T}} : \mathbf{C}_e}{\rho_R} = 2\frac{\tilde{\mathbf{T}} : \mathbf{E}_e}{\rho_R} + \frac{\mathrm{tr}\tilde{\mathbf{T}}}{\rho_R}, \tag{4.101}$$

which imply

$$\frac{p}{\rho} = \frac{\tilde{p}}{\rho_R} - \frac{2}{3}\frac{\tilde{\mathbf{T}} : \mathbf{E}_e}{\rho_R}. \tag{4.102}$$

For the special case where the internal variable q_1 is taken to be ρ_R, we see from $(4.77)_3$ and (4.91) and (4.92) that the plastic dissipation inequality takes the form

$$\frac{1}{\rho_R}\sum_{n=2}^{N}Q_n\dot{q}_n + \frac{1}{\rho_R}\sum_{k=1}^{K}\mathbf{\Lambda}_k : \dot{\mathbf{A}}_k \leqslant \mathcal{P}_p^s + \left(\frac{p}{\rho} - Q_1\right)\frac{\dot{\rho}_R}{\rho_R}$$
$$= \frac{1}{\rho}\left[\mathbf{S} : \mathbf{D}_p + (p - \rho Q_1)\frac{\dot{\rho}_R}{\rho_R}\right]. \tag{4.103}$$

The same grouping of terms can be used in the energy balance Eqs. (4.83) and (4.84). Recall that Q_n and $\mathbf{\Lambda}_k$ are given by (4.75) and (4.76). In particular, when $q_1 = \rho_R$ we have

$$Q_1 = \rho_R\left(\frac{\partial\psi}{\partial\rho_R}\right)_\theta = \rho_R\left(\frac{\partial e}{\partial\rho_R}\right)_\eta = -\theta\rho_R\left(\frac{\partial\eta}{\partial\rho_R}\right)_e. \tag{4.104}$$

The combination of terms $p - \rho Q_1$ in (4.103) represents an effective pressure that

does work against changing intermediate density. This effective pressure consists essentially of the difference between the usual external pressure and the internal force conjugate to the intermediate density. The existence of an effective pressure may have some bearing on the curious fact that metals that display a stress differential effect (different yield stresses in tension and compression) do not show the corresponding volumetric change that might have been expected from the normality rule (Spitzig et al., 1975, 1976).

4.9. The Gibbs function

In plasticity theories, it is common to take stress, rather than elastic strain, as the primary mechanical variable. Although the Cauchy stress tensor $\mathbf{T}$ is a natural choice, adopting $\mathbf{T}$ as the primary mechanical variable does not lead to a theory equivalent to the one outlined above if the material has anistropic elastic response. In this subsection we assume that the constitutive relation (4.81) for $\tilde{\mathbf{T}}$ is invertible in $\mathbf{E}_e$ for fixed θ, q_n, and $\mathbf{A}_k$, which also implies fixed ρ_R since $\rho_R = \hat{\rho}_R(q_n, \mathbf{A}_k)$. Thus

$$E_e = \mathcal{E}\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right),\tag{4.105}$$

and any constitutive relations involving $\mathbf{E}_e$ may be rewritten in terms of $\tilde{\mathbf{T}}$. In this case it is useful to introduce the Gibbs function (per unit mass), g, defined by

$$g = -\psi + \frac{\tilde{\mathbf{T}} : \mathbf{E}_e}{\rho_R} = -e + \theta\eta + \frac{\tilde{\mathbf{T}} : \mathbf{E}_e}{\rho_R}.\tag{4.106}$$

In view of (4.105) and the fact that $\rho_R = \hat{\rho}_R(q_n, \mathbf{A}_k)$, g may be regarded as a function of $\tilde{\mathbf{T}}$, θ, q_n, and $\mathbf{A}_k$,

$$g = \tilde{g}\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right).\tag{4.107}$$

Eq. $(4.106)_1$ and the potential relations (4.69) imply that g is a potential for the elastic strain and the entropy,

$$\mathbf{E}_e = \rho_R \frac{\partial g}{\partial \tilde{\mathbf{T}}}\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right), \quad \eta = \frac{\partial g}{\partial \theta}\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right).\tag{4.108}$$

From $(4.106)_1$, the expression $(4.20)_1$ for $\dot{\mathcal{W}}_s$, and the relation $\mathcal{P}_e = \left(\tilde{\mathbf{T}}/\rho_e\right) : \dot{\mathbf{E}}_e$, we find that the rate of change of the stored energy of cold work is given by

$$\dot{\mathcal{W}}_s = -\dot{g} + \eta\dot{\theta} + \mathbf{E}_e : \left(\tilde{\mathbf{T}}/\rho_R\right)^{\cdot}$$

$$= -\sum_{n=1}^{N} \frac{\partial g}{\partial q_n} \dot{q}_n - \sum_{k=1}^{K} \frac{\partial q}{\partial \mathbf{A}_k} : \dot{\mathbf{A}}_k - \frac{\tilde{\mathbf{T}} : \mathbf{E}_e}{\rho_R} \frac{\dot{\rho}_R}{\rho_R},\tag{4.109}$$

where the second equation follows from the potential relations (4.108).[18] When the relation (4.89) for $\dot{\rho}_R/\rho_R$ is substituted into (4.109)$_2$, we find that the conjugate internal forces Q_n and Λ_k [see (4.75)–(4.77)] are also given by

$$Q_n = -\rho_R \frac{\partial g}{\partial q_n} - \frac{\tilde{\mathbf{T}} : \mathbf{E}_e}{\rho_R} \frac{\partial \rho_R}{\partial q_n}, \quad \Lambda_k = -\rho_R \frac{\partial g}{\partial \mathbf{A}_k} - \frac{\tilde{\mathbf{T}} : \mathbf{E}_e}{\rho_R} \frac{\partial \rho_R}{\partial \mathbf{A}_k}. \tag{4.110}$$

From (4.101)–(4.102) it follows that the term $\tilde{\mathbf{T}}/\rho_R : \mathbf{E}_e$ in the above relations may be expressed as[19]

$$\frac{\tilde{\mathbf{T}} : \mathbf{E}_e}{\rho_R} = \frac{1}{2}\left(\frac{\mathrm{tr}\mathbf{T}}{\rho} - \frac{\mathrm{tr}\tilde{\mathbf{T}}}{\rho_R}\right) = \frac{3}{2}\left(\frac{\tilde{p}}{\rho_R} - \frac{p}{\rho}\right). \tag{4.111}$$

For the special case where ρ_R is itself an internal variable, say $\rho_R = q_1$, the expressions (4.110) for the conjugate internal forces reduce to

$$Q_1 = -\rho_R \frac{\partial g}{\partial \rho_R} - \frac{\tilde{\mathbf{T}} : \mathbf{E}_e}{\rho_R}, \quad Q_n = -\rho_R \frac{\partial g}{\partial q_n} \, (n \geqslant 2), \quad \Lambda_k - \rho_R \frac{\partial q}{\partial \mathbf{A}_k}. \tag{4.112}$$

In this case, the plastic dissipation inequality takes the form (4.103).

4.10. Generalized stress and strain tensors

Most of the results in this section have been expressed in terms of the elastic strain tensor $\mathbf{E}_e$ and second Piola–Kirchhoff stress tensor $\tilde{\mathbf{T}}$ relative to the intermediate configuration. For example, we have shown that the elastic–plastic decomposition of the stress power is given by $\mathcal{P} = \mathcal{P}_e + \mathcal{P}_p$, where $\rho_R \mathcal{P}_e = \tilde{\mathbf{T}} : \dot{\mathbf{E}}_e$ and $\rho_R \mathcal{P}_p = \tilde{\mathbf{T}} : \tilde{\mathbf{D}}_p$. In view of this expression for the elastic stress power, the elastic strain tensor $\mathbf{E}_e$ is said to be (work) conjugate to the stress tensor $\tilde{\mathbf{T}}$. Of course, other finite elastic strain tensors may also be used, in which case their conjugate stress tensors are of interest.

Following Hill (1978),[20] to any smooth real valued function f defined on the positive reals and satisfying the conditions $f(1) = 0$, $f'(1) = 1$, and $f' > 0$, we may associate a tensor valued function $\mathbf{f}$ that maps the set of symmetric positive-definite

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[18] An alternate approach is to regard $\tilde{\mathbf{T}}$, rather than $\mathbf{E}_e$, as the primary mechanical variable from the start. Then (4.105), (4.107), and an analogous relation for η are taken as the (properly invariant) thermoelastic constitutive assumptions. The entropy inequality becomes (4.21) with $\mathcal{W}_s$ given by (4.109)$_1$. On expanding the $\dot{g}$ term in (4.109)$_1$, we see that $\mathcal{W}_s$ is equal to $\left(\mathbf{E}_e/\rho_R - \partial g/\partial\tilde{\mathbf{T}}\right) : \dot{\tilde{\mathbf{T}}} + (\eta - \partial g/\partial\theta)\dot{\theta}$ plus the terms on the second line of (4.109). If $\mathbf{F}$ is replaced by $\tilde{\mathbf{T}}$ in the definition of instantaneous thermoelastic response in Section 4.4, and if it is assumed that $\tilde{\mathbf{T}}$ and θ may be prescribed arbitrarily, then the entropy inequality is satisfied iff the potential relations (4.108) hold, in which case $\mathcal{W}_s$ reduces to the expression on the second line of (4.109). Finally, if the relation (4.105) for $\mathbf{E}_e$ is assumed to be invertible in $\tilde{\mathbf{T}}$, we obtain the potential relations (4.69).

[19] Also note that $\left(\tilde{\mathbf{T}}/\rho_R\right) : \mathbf{E}_e = (\mathbf{T}/\rho) : \frac{1}{2}(\mathbf{I} - \mathbf{B}_e^{-1})$, where $\mathbf{B}_e = \mathbf{F}_e\mathbf{F}_e^T = \mathbf{V}_e^2$.

[20] See also Ogden (1984) and Scheidler (1991). All of these authors considered total strain tensors rather than their elastic part as considered here, but the ideas are essentially the same in either case.

tensors one-to-one into the space of symmetric tensors. If the symmetric positive-definite tensor $\mathbf{A}$ has eigenvalues a_i, then $\mathbf{f}(\mathbf{A})$ is defined to be the symmetric tensor that is coaxial with $\mathbf{A}$ but with corresponding eigenvalues $f(a_i)$. In particular, the elastic strain tensor $\mathbf{E}_e^f$ corresponding to the function f is defined by $\mathbf{E}_e^f = \mathbf{f}(\mathbf{U}_e)$.

The functions $f_0(\lambda) = \ln\lambda$ and $f_m(\lambda) = \frac{1}{m}(\lambda^m - 1)$ for any nonzero real m satisfy the above conditions. This one-parameter family of functions generates a one-parameter family of elastic strain tensors $\mathbf{E}_e^{f_m}$, which will also be denoted simply by $\mathbf{E}_e^{(m)}$:

$$\mathbf{E}_e^{(m)} = \frac{1}{m}\left(\mathbf{U}_e^m - \mathbf{I}\right) \quad (m \neq 0), \quad \mathbf{E}_e^{(0)} = \ln(\mathbf{U}_e). \tag{4.113}$$

In particular, $\mathbf{E}_e^{(2)} = \frac{1}{2}\left(\mathbf{U}_e^2 - \mathbf{I}\right) = \mathbf{E}_e$.

Again following Hill (1978), the (symmetric) stress tensor $\tilde{\mathbf{T}}_f$ conjugate to the elastic strain tensor $\mathbf{E}_e^f$ is defined by the condition that

$$\rho_R \mathcal{P}_e = \tilde{\mathbf{T}}_f : \dot{\mathbf{E}}_e^f \tag{4.114}$$

for all motions; equivalently,

$$\tilde{\mathbf{T}}_f : \dot{\mathbf{E}}_e^f = \tilde{\mathbf{T}} : \dot{\mathbf{E}}_e. \tag{4.115}$$

For the special case $f = f_m$, the stress tensor conjugate to $\mathbf{E}_e^{(m)}$ is also denoted by $\tilde{\mathbf{T}}_{(m)}$. Note that $\tilde{\mathbf{T}}_{(2)} = \tilde{\mathbf{T}}$.

Let $D\mathbf{f}(\mathbf{U}_e)$ denote the derivative of $\mathbf{f}$ evaluated at $\mathbf{U}_e$. The fourth-order tensor $D\mathbf{f}(\mathbf{U}_e)$, regarded as a linear transformation on the space of symmetric tensors, is symmetric and nonsingular. Thus,

$$\tilde{\mathbf{T}}_f : \dot{\mathbf{E}}_e^f = \tilde{\mathbf{T}}_f : D\mathbf{f}(\mathbf{U}_e)\left[\dot{\mathbf{U}}_e\right] = D\mathbf{f}(\mathbf{U}_e)\left[\tilde{\mathbf{T}}_f\right] : \dot{\mathbf{U}}_e,$$

and

$$\tilde{\mathbf{T}} : \dot{\mathbf{E}}_e = \tilde{\mathbf{T}} : \frac{1}{2}\left(\mathbf{U}_e\dot{\mathbf{U}}_e + \dot{\mathbf{U}}_e\mathbf{U}_e\right) = \frac{1}{2}\left(\mathbf{U}_e\tilde{\mathbf{T}} + \tilde{\mathbf{T}}\mathbf{U}_e\right) : \dot{\mathbf{U}}_e = \mathrm{sym}\left(\mathbf{U}_e\tilde{\mathbf{T}}\right) : \dot{\mathbf{U}}_e,$$

so (4.115) implies that

$$\left\{D\mathbf{f}(\mathbf{U}_e)\left[\tilde{\mathbf{T}}_f\right] - \mathrm{sym}\left(\mathbf{U}_e\tilde{\mathbf{T}}\right)\right\} : \dot{\mathbf{U}}_e = 0$$

for arbitrary values of $\dot{\mathbf{U}}_e$. The expression in braces is necessarily zero provided that it is independent of $\dot{\mathbf{U}}_e$. Since $\tilde{\mathbf{T}}$ is independent of $\dot{\mathbf{U}}_e$, we need only add the natural assumption that $\tilde{\mathbf{T}}_f$ is independent of $\dot{\mathbf{U}}_e$. Then $D\mathbf{f}(\mathbf{U}_e)[\tilde{\mathbf{T}}_f] = \mathrm{sym}(\mathbf{U}_e\tilde{\mathbf{T}})$, and the stress tensor conjugate to the elastic strain tensor $\mathbf{E}_e^f$ is given by

$$\tilde{T}_f = D\mathbf{f}(\mathbf{U}_e)^{-1}\left[\operatorname{sym}\left(\mathbf{U}_e\tilde{T}\right)\right] \equiv \mathbb{F}_{\mathbf{U}_e}\left[\tilde{T}\right]. \tag{4.116}$$

The fourth-order tensor $\mathbb{F}_{\mathbf{U}_e}$, regarded as a linear transformation on the space of symmetric tensors, is symmetric and nonsingular.

Since $\mathbf{f}$ is one-to-one, we may express $\mathbf{E}_e$ in terms of any elastic strain tensor $\mathbf{E}_e^f$. In view of (4.114), the potential relations (4.69), (4.72), and (4.74) for the stress hold with $\tilde{T}$ replaced by $\tilde{T}_f$ and $\mathbf{E}_e$ replaced by $\mathbf{E}_e^f$, for example, $\tilde{T}_f = \partial\psi/\partial\mathbf{E}_e^f$. If this can be solved for $\mathbf{E}_e^f$ in terms of $\tilde{T}_f$, then we may define a new Gibbs function g_f by replacing $\tilde{T} : \mathbf{E}_e$ with $\tilde{T}_f : \mathbf{E}_e^f$ in (4.106) (note that $\tilde{T} : \mathbf{E}_e \neq \tilde{T}_f : \mathbf{E}_e^f$ in general). Then the relations (4.107)–(4.110) and (4.112) hold with $\tilde{T}$, $\mathbf{E}_e$, and g replaced by $\tilde{T}_f$, $\mathbf{E}_e^f$, and g_f, respectively. The identity (4.111)$_1$ generalizes to

$$m\frac{\tilde{T}_{(m)} : \mathbf{E}_e^{(m)}}{\rho_R} = \frac{\operatorname{tr}\mathbf{T}}{\rho} - \frac{\operatorname{tr}\mathbf{T}_{(m)}}{\rho_R}. \tag{4.117}$$

In particular, for $m = 0$ this reduces to

$$\frac{\operatorname{tr}\tilde{T}_{(0)}}{\rho_R} = \frac{\operatorname{tr}\mathbf{T}}{\rho}, \tag{4.118}$$

where $\tilde{T}_{(0)}$ is the stress tensor conjugate to the logarithmic elastic strain tensor $\mathbf{E}_e^{(0)} = \ln(\mathbf{U}_e)$.

Since $\rho_R\mathcal{P}_p = \tilde{T} : \tilde{D}_p$, we may regard $\tilde{D}_p$ as the (total) plastic strain rate tensor conjugate to the stress tensor $\tilde{T}$. Then the (total) plastic strain rate tensor $\tilde{D}_p^f$ conjugate to the stress tensor $\tilde{T}_f$ should satisfy the condition $\tilde{T}_f : \tilde{D}_p^f = \tilde{T} : \tilde{D}_p$. Actually, the plastic shearing tensor $\tilde{D}_p$ will be of more interest in the remainder of the paper. Recall from Section 4.8 that the contribution of the plastic shearing to the plastic stress power $\mathcal{P}_p$ is given by $\mathcal{P}_p^s$, where $\rho_R\mathcal{P}_p^s = \tilde{T} : \tilde{D}_p$, so that we may refer to $\tilde{D}_p$ as the plastic shearing tensor conjugate to the stress tensor $\tilde{T}$. Then the (symmetric) plastic shearing tensor $\tilde{D}_p^f$ conjugate to the stress tensor $\tilde{T}_f$ should satisfy the condition

$$\rho_R\mathcal{P}_p^s = \tilde{T}_f : \tilde{D}_p^f; \tag{4.119}$$

equivalently,

$$\tilde{T} : \tilde{D}_p = \tilde{T}_f : \tilde{D}_p^f. \tag{4.120}$$

On using the relation (4.116) for $\tilde{T}_f$, we see that (4.120) implies

$$\tilde{T} : \tilde{D}_p = \mathbb{F}_{\mathbf{U}_e}\left[\tilde{T}\right] : \tilde{D}_p^f = \tilde{T} : \mathbb{F}_{\mathbf{U}_e}\left[\tilde{D}_p^f\right]. \tag{4.121}$$

This will hold for all values of $\tilde{T}$ if we take[21]

$$\tilde{D}_p = \mathbb{F}_{U_e}\left[\tilde{D}_p^f\right], \quad \tilde{D}_p^f = \mathbb{F}_{U_e}^{-1}\left[\tilde{D}_p\right]. \tag{4.122}$$

Recalling that $\tilde{D}_p = \text{sym}\left(C_e\dot{F}_pF_p^{-1}\right)$, we see that the plastic shearing tensor $\tilde{D}_p^f$ conjugate to the stress tensor $\tilde{T}_f$ depends only on U_e and $\tilde{D}_p$, or equivalently, on U_e and $\dot{F}_pF_p^{-1}$.[22]

5. The evolution equations and yield function

The thermodynamic restrictions derived in Section 4 hold for the general class of materials with instantaneous thermoelastic response, as defined in Section 4.4. In the remainder of the paper we focus on materials that satisfy evolution equations of the general form (4.54)–(4.56), which we repeat here:

$$\dot{F}_p = \mathcal{H}(F_e, \theta, q_n, A_k, F_p), \tag{5.1}$$

$$\dot{q}_m = \xi_m(F_e, \theta, q_n, A_k, F_p), \tag{5.2}$$

$$\dot{A}_j = \mathcal{A}_j(F_e, \theta, q_n, A_k, F_p), \tag{5.3}$$

for $m, n = 1, \ldots, N$ and $j, k = 1, \ldots, K$. Of course, we continue to assume that the thermoelastic constitutive relations in Section 4.3 hold. Then as shown in Section 4.4, these materials have instantaneous thermoelastic response provided the constitutive functions $\mathcal{H}$, ξ_m, and $\mathcal{A}_j$, are Lipschitz continuous on any compact subset of their domain. In Section 5.1 additional restrictions on these constitutive functions are deduced or imposed. Of course, the property of instantaneous thermoelastic response continues to hold when these restrictions are made, so the subsequent constitutive theory will be complete in the sense discussed in Section 4.4. In Section 5.2 we consider transformations to new sets of internal variables. Yield surfaces and conditions for the onset of the evolution of the internal variables are discussed in Section 5.3.

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[21] Note that (4.122)$_1$ is sufficient but not necessary for (4.121) to hold for all values of $\tilde{T}$, since $\tilde{D}_p$ might not be independent of $\tilde{T}$. Indeed, flow rules for which $\tilde{D}_p$ depends explicitly on $\tilde{T}$ are of particular interest; see Section 5.1.

[22] Except for the special cases corresponding to $E_e^f = E_e^{(m)}$ with m an integer, explicit component-free formulas for the above results are rather complicated. In general, it is simpler to express these results in component form relative to a principal basis for U_e. Component formulas for $Df(U_e)$ can be found in Ogden (1984) and Scheidler (1991).

5.1. Restrictions on the evolution equations

In Section 4.7, it was shown that for a material with instantaneous thermoelastic response, the entropy inequality implies that ρ_R can depend only on the current values of the internal state variables. This result and the relation (3.13) between $\mathbf{F}_p$, $\mathbf{F}_p$, and ρ_R are summarized here:

$$\mathbf{F}_p = \left(\frac{\rho_0}{\rho_R}\right)^{1/3} \mathsf{F}_n, \quad \rho_R = \hat{\rho}_R(q_n, \mathbf{A}_k). \tag{5.4}$$

Clearly, we are only free to prescribe an evolution equation for the isochoric part F_p of $\mathbf{F}_p$, say $\dot{\mathsf{F}}_p = \bar{\mathcal{H}}(\mathbf{F}_e, \theta, q_n, \mathbf{A}_k, \mathsf{F}_p)$. In view of (5.4), any dependence on the dilational part of $\mathbf{F}_p$ may be absorbed into the internal variables, so that $\dot{\mathsf{F}}_p = \hat{\mathcal{H}}(\mathbf{F}_e, \theta, q_n, \mathbf{A}_k, \mathsf{F}_p)$. However, it is more convenient to express such a relation in an equivalent form in terms of the plastic velocity gradient[23] $\dot{\mathsf{F}}_p \mathsf{F}_p^{-1}$, say

$$\dot{\mathsf{F}}_p \mathsf{F}_p^{-1} = \mathcal{F}_p(\mathbf{F}_e, \theta, q_n, \mathbf{A}_k, \mathsf{F}_p). \tag{5.5}$$

Since $\mathrm{tr}\left(\dot{\mathsf{F}}_p \mathsf{F}_p^{-1}\right) = 0$, the values of the flow function $\mathcal{F}_p$ are deviatoric tensors. If desired, we may use (5.5), (5.4)$_2$, and the decomposition (4.87)$_1$ of $\dot{\mathsf{F}}_p \mathsf{F}_p^{-1}$ to obtain an evolution equation for $\mathbf{F}_p$ of the general form (5.1), but it is simpler to work directly with (5.5) and (5.4).

The evolution Eq. (5.5) for F_p is known as the flow rule. Recall that F_p is invariant under superposed rigid motions, and hence, so is $\dot{\mathsf{F}}_p \mathsf{F}_p^{-1}$. Arguing as in Section 4.3, we find that the flow rule (5.5) is properly invariant iff the flow function $\mathcal{F}_p$ does not depend on the elastic rotation tensor $\mathbf{R}_e$, so that

$$\dot{\mathsf{F}}_p \mathsf{F}_p^{-1} = \mathcal{F}_p(\mathbf{U}_e, \theta, q_n, \mathbf{A}_k, \mathsf{F}_p) = \hat{\mathcal{F}}_p(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k, \mathsf{F}_p). \tag{5.6}$$

In plasticity theories, it is common to assume that the flow rule can be expressed in terms of the stress, say

$$\dot{\mathsf{F}}_p \mathsf{F}_p^{-1} = \tilde{\mathcal{F}}_p\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k, \mathsf{F}_p\right). \tag{5.7}$$

Since $\tilde{\mathbf{T}} = \tilde{\tau}(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k)$, we see that (5.7) is a special case of (5.6). If the stress–strain relation is invertible in $\mathbf{E}_e$, then (5.7) is equivalent to (5.6). *Henceforth, we assume that the stress–strain relation $\tilde{\mathbf{T}} = \tilde{\tau}(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k)$ is invertible in $\mathbf{E}_e$.* Then $\tilde{\mathbf{T}}$ may be used in place of $\mathbf{E}_e$ as the primary mechanical variable.

By applying some of the above arguments to the evolution Eqs. (5.2) and (5.3) for the internal variables, we see that properly invariant forms of these equations are

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[23] This term is motivated by the fact that the analogous expression for the total deformation gradient, namely $\dot{\mathbf{F}}\mathbf{F}^{-1}$, is equal to the spatial velocity gradient. However, $\dot{\mathsf{F}}_p \mathsf{F}_p^{-1}$ need not be the gradient of any vector field.

$\dot{q}_m = \hat{\xi}_m(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k, \mathbf{F}_p)$ and $\dot{\mathbf{A}}_j = \hat{\mathcal{A}}_j(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k, \mathbf{F}_p)$. We now assume that $\dot{q}_m$ and $\dot{\mathbf{A}}_j$ are independent of $\mathbf{F}_p$, so that

$$\dot{q}_m = \hat{\xi}_m(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k) = \tilde{\xi}_m\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right), \tag{5.8}$$

$$\dot{\mathbf{A}}_j = \hat{\mathcal{A}}_j(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k) = \tilde{\mathcal{A}}_j\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right). \tag{5.9}$$

This assumption may be physically motivated as follows. We regard the state of the material in the intermediate configuration as being characterized solely by the current values of the internal state variables q_n and $\mathbf{A}_k$. The state of the current configuration is characterized by these internal variables and the thermomechanical variables $\mathbf{F}_e$ and θ, which describe the thermoelastic deformation from the intermediate configuration to the current configuration. $\mathbf{F}_p$ represents the part of the total deformation gradient arising from plastic slip, which does not distort the crystal lattice except in the neighborhood of dislocations and other defects. Since the effects of these short range disruptions in the lattice are, by assumption, accounted for by the internal variables, $\mathbf{F}_p$ does not measure any aspect of the state of the material even though plastic slip is the fundamental process that generates changes in the internal variables. Thus $\mathbf{F}_p$ should not be regarded as an internal state variable, nor should any function of $\mathbf{F}_p$, such as the plastic strain tensor $\frac{1}{2}\left(\mathbf{F}_p^T \mathbf{F}_p - \mathbf{I}\right)$. The constitutive relations (4.34)–(4.36) for ψ, e, and $\mathbf{T}$ reflect the assumption that these variables should depend only on the current state of the material. Likewise, the evolution Eqs. (5.8) and (5.9) reflect the assumption that the rate of change of the state of the intermediate configuration should depend only on the current state. These assumptions seem to reflect the views of many material scientists and constitutive modelers (Teodosiu and Sidoroff, 1976; Davison et al., 1977; Anand, 1985; Kocks, 1987; Cleja-Ţigoiu and Soós, 1990).

An elementary thought experiment also reveals why $\mathbf{F}_p$ would make a poor choice for an internal state variable. Consider a simple tension/compression test on a metal rod. Initially, $\mathbf{F} = \mathbf{F}_p = \mathbf{I}$. As the rod is pulled in tension beyond the elastic range, it is reasonable to expect that $\|\mathbf{F}_p - \mathbf{I}\|$ increases with the length of the rod. If the rod is then compressed back to its original length, the final values of $\mathbf{F}$ and $\mathbf{F}_p$ should be close to their initial value $\mathbf{I}$. Plastic deformation occurs during both stages of this process, and the dislocation density would be expected to increase during both stages as well, but this is not reflected in the final value of $\mathbf{F}_p$.

The exclusion of $\mathbf{F}_p$ as an internal state variable does not exclude the use of an "equivalent plastic strain," say ε_p, as a scalar internal variable.[24] Such variables are typically defined by an evolution equation of the form $\dot{\varepsilon}_p = c\|\mathbb{D}_p\|$, where the symmetric tensor $\mathbb{D}_p$ is some measure of plastic strain rate and c is a positive constant. If the flow rule implies that $\mathbb{D}_p$ is a function of $\mathbf{E}_e$, θ, q_n, and $\mathbf{A}_k$, then this evolution equation for ε_p is of the general form (5.8). Note that ε_p is a nondecreasing function

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[24] This is not to say that we regard an equivalent plastic strain as an accurate measure of any aspect of the state of the material, but only that such internal variables fit within our framework.

of time since $\dot{\varepsilon}_p \geqslant 0$. The objection against F_p as an internal variable in the thought experiment above does not apply to ε_p, since ε_p would increase during both the tensile and compressive stages of plastic deformation. Note that the value of ε_p is path dependent, so ε_p is not a true strain measure.

Next, observe that if any one of the variables δ, Δ, $\mathcal{P}_p$, $\mathcal{P}_p^s$, or $\rho_R \mathcal{P}_p^s$, depends only on the current state, then all of them do. Since these are scalar variables, such dependence must be through the reduced state variables $(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k)$ or $(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k)$. To prove the observation, first observe that when the evolution Eqs. (5.8) and (5.9) for the internal variables and the constitutive relations (4.45) and (4.46) for ψ and e are substituted into the expressions (4.77) for $\dot{\mathcal{W}}_s$, we see that $\dot{\mathcal{W}}_s$ depends only on the state variables. In other words, the rate of storage of cold work depends only on the current state, a result that is consistent with the views expressed above. Since the internal dissipation δ is given by $\mathcal{P}_p - \dot{\mathcal{W}}_s$ [see (4.19)], it follows that δ depends only on the current state iff the plastic stress power $\mathcal{P}_p$ does. Next, recall that $\mathcal{P}_p$ may be decomposed into a contribution $\mathcal{P}_p^s$ from dislocation slip and a contribution $\mathcal{P}_p^v$ from plastic volume change (see Section 4.8). Now by (4.89), (4.80), and (5.8)–(5.9), $\dot{\rho}_R / \rho_R$ depends only on the state variables, so from (4.101) and the expression (4.92) for $\mathcal{P}_p^v$, we see that $\mathcal{P}_p^v$ depends only on the state variables. Therefore, $\mathcal{P}_p$ depends only on the current state iff $\mathcal{P}_p^s$ does. Also, since ρ_R depends only on the internal variables or is itself an internal variable, it follows that $\mathcal{P}_p^s$ depends only on the current state iff $\rho_R \mathcal{P}_p^s$ does. Note that $\rho_R \mathcal{P}_p^s$ represents the plastic stress power per unit volume in the intermediate configuration due to dislocation slip. Finally, by (4.75)–(4.76) and (5.8)–(5.9), we see that the sums in (4.85)$_1$ depend only on the current state. Thus the inelastic heating Δ depends only on the current state iff $\mathcal{P}_p$ does.

As noted above, the evolution Eqs. (5.8) and (5.9) for the internal variables reflect the idea that the rate of change of the state of the intermediate configuration should depend only on the current state of the material. For similar reasons, we argue that "the rate of plastic deformation" should also depend only on the current state. A precise formulation of this property is complicated by the fact that it is not invariant under changes in the measure of plastic deformation rate.[25] For example, both $\dot{F}_p$ and $\dot{F}_p F_p^{-1}$ measure the rate of plastic deformation, but if we assume that one of these rates depends only on the state variables, then the other necessarily depends not only on the state variables but also on F_p. Therefore the flow rule actually consists of two parts. First, there is the selection of some preferred measure of the rate of plastic deformation. Second, there is the choice of the particular flow function, which is assumed to depend only on the state variables.

In the slip theory for a single crystal or polycrystalline grain, $\dot{F}_p F_p^{-1}$ is expressed in terms of the shearing rates on the active slip systems, the slip directions, and the normals to the slip planes (Teodosiu and Sidoroff, 1976; Davison et al., 1977; Asaro, 1983; Obata et al., 1990). In this theory the plastic velocity gradient $\dot{F}_p F_p^{-1}$ clearly

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[25] Likewise, the question of the invariance of the form of the evolution equations under transformations of the internal variables needs to be addressed. This is taken up in the next subsection.

enjoys a preferred status over other tensor measures of the rate of plastic deformation. This suggests that we consider flow rules of the form

$$\dot{F}_p F_p^{-1} = \hat{\mathcal{F}}_p(E_e, \theta, q_n, A_k) = \tilde{\mathcal{F}}_p\left(\tilde{T}, \theta, q_n, A_k\right), \tag{5.10}$$

i.e. (5.6) and (5.7) with F_p eliminated from the list of arguments. For this class of flow rules, not only does the preferred measure of the rate of plastic deformation, namely the plastic velocity gradient $\dot{F}_p F_p^{-1}$, depend only on the state variables, but so does the stress power due to dislocation slip. Indeed, recall that by (4.99) and (4.100), we have

$$\rho_R \mathcal{P}_p^s = \tilde{T} : \tilde{D}_p = C_e \tilde{T} : \dot{F}_p F_p^{-1}. \tag{5.11}$$

Thus, when the flow rule (5.10) holds, we see that $\rho_R \mathcal{P}_p^s$ depends only on the state variables.

Flow rules of the form (5.10) are commonly used not only for single crystals and polycrystalline grains but also in continuum models of polycrystals. In a continuum model of a polycrystal, a representative volume element must contain many grains. Any measure of rate of plastic deformation at a continuum point represents an appropriate average over these grains. Furthermore, the *effects* of grain boundaries must be accounted for, even if the boundaries themselves are not modeled explicitly. In this case it is not obvious that (5.10) is necessarily the most appropriate form for the flow rule, and we are led to at least consider other possibilities. In doing so we are guided by three criteria. First, as discussed above, the flow rule should reflect the idea that some preferred measure of the rate of plastic deformation depends only on the state variables. Second, even though we wish to consider measures other than $\dot{F}_p F_p^{-1}$, the flow rule should ultimately be expressible in the form (5.6) or (5.7), though not necessarily in the simpler form (5.10). Third, we wish to retain one of the fundamental properties of the flow rules (5.10), namely, that the stress power due to dislocation slip ($\mathcal{P}_p^s$ or $\rho_R \mathcal{P}_p^s$) depends only on the state variables. As observed above, this implies that the internal dissipation δ and the inelastic heating Δ depend only on the state variables.

This third property does not follow from any of the thermodynamic restrictions derived in Section 4; and in view of (5.11), it need not hold for all flow rules of the form (5.6) and (5.7) because of the possible dependence of $\dot{F}_p F_p^{-1}$ on F_p. On the other hand, from (5.11) we immediately see that if the plastic shearing tensor $\tilde{D}_p$ depends only on the state variables, say

$$\tilde{D}_p = \hat{\mathcal{D}}_p(E_e, \theta, q_n, A_k) = \tilde{\mathcal{D}}_p\left(\tilde{T}, \theta, q_n, A_k\right), \tag{5.12}$$

then so does $\rho_R \mathcal{P}_p^s$. Note that while either (5.10) or (5.12) is sufficient for the stress power due to dislocation slip to depend only on the state variables, neither relation is necessary. Indeed, to the right-hand side of (5.10) and (5.12) we could add any

functions of $\tilde{T}$, θ, q_n, $\mathbf{A}_k$, and $\mathbf{F}_p$ that are orthogonal to $\mathbf{C}_e\tilde{T}$ and $\tilde{T}$, respectively. However, such terms are difficult to motivate, and in any case they would seem to violate the first criterion mentioned above.

Recall that, in Section 4.10, we introduced the generalized elastic strain tensors $\mathbf{E}_e^f$, the conjugate stress tensors $\tilde{T}_f$, and their conjugate plastic shearing tensors $\tilde{D}_p^f$. Since $\rho_R \mathcal{P}_p^s = \tilde{T} : \tilde{D}_p = \tilde{T}_f : \tilde{D}_p^f$, it follows that if

$$\tilde{D}_p^f = \hat{\mathcal{D}}_p^f(\mathbf{E}_e^f, \theta, q_n, \mathbf{A}_k) = \check{\mathcal{D}}_p^f\left(\tilde{T}_f, \theta, q_n, \mathbf{A}_k\right), \tag{5.13}$$

then $\rho_R \mathcal{P}_p^s$ depends only on the state variables. Note that the relations (5.12) and (5.13) are equivalent in the sense that any relation of the form (5.12) can be expressed in the form (5.13) and vice versa. This follows from (4.116), (4.122) and the fact that $\mathbf{U}_e$ is a state variable.

If $\mathbf{C}_e$ and $\tilde{T}$ commute, then $\mathbf{C}_e\tilde{T}$ is a symmetric tensor, so in the inner product $(5.11)_2$ we may replace $\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}$ by its symmetric part:[26]

$$\rho_R \mathcal{P}_p^s = \mathbf{C}_e\tilde{T} : \mathrm{sym}\left(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}\right), \quad \text{if } \mathbf{C}_e\tilde{T} = \tilde{T}\mathbf{C}_e. \tag{5.14}$$

Although $\mathbf{C}_e$ and $\tilde{T}$ need not commute in general, they do commute if the material is elastically isotropic, that is, if

$$\tilde{T} = \tilde{\tau}(\mathbf{E}_e, \theta, q_n) \tag{5.15}$$

for some isotropic function $\tilde{\tau}$.[27] From (5.14) it follows that for an elastically isotropic material a sufficient (but not necessary) condition for the stress power due to dislocation slip to depend only on the state variables is that $\mathrm{sym}\left(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}\right)$ depend only on the state variables, say

$$\mathrm{sym}\left(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}\right) = \hat{\mathcal{G}}_p(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k) = \check{\mathcal{G}}_p\left(\tilde{T}, \theta, q_n, \mathbf{A}_k\right). \tag{5.16}$$

Observe that the flow rule (5.10) implies (5.16) as well as the relation (5.12) [since $\tilde{D}_p = \mathrm{sym}(\mathbf{C}_e\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})$], whereas neither (5.16) nor (5.12) is sufficient to determine $\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}$. To obtain a complete flow rule, (5.12) or (5.16) must be supplemented by an additional constitutive relation for some tensor measure of plastic spin. We return to this topic in Section 6.

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[26] Note that the relations (5.11) and (5.14) for $\rho_R \mathcal{P}_p^s$ remain valid if $\tilde{T}$ is replaced with $\tilde{S}$ (see Section 4.8).

[27] Using (3.16) it is easily shown that an equivalent relation in terms of the Cauchy stress is $T = \hat{\tau}(\mathbf{V}_e, \theta, q_n)$ or some isotropic function $\hat{\tau}$. This in turn is equivalent to the condition that $T = \hat{\tau}(\mathbf{F}_e, \theta, q_n) = \hat{\tau}(\mathbf{F}_e\mathbf{H}, \theta, q_n)$ for any rotation $\mathbf{H}$.

5.2. Transformations of the internal variables

We have shown that ρ_R can depend at most on the internal variables. As observed previously, this result includes the special case where ρ_R is itself an internal variable, say $\rho_R = q_1$. We might also have the situation where $\rho_R = \hat{\rho}_R(q_n, \mathbf{A}_k) \neq q_1$, but the function ρ_R is invertible in q_1, in which case we can solve for $q_1 = \hat{q}_1(\rho_R, q_2, \ldots, q_N, \mathbf{A}_k)$. Then the internal variable q_1 can be replaced with ρ_R. Similarly, consider a scalar q that depends only on the internal variables, so that

$$q = \hat{q}(q_n, \mathbf{A}_k). \tag{5.17}$$

Then from (5.17), (5.8), and (5.9), we have

$$
\begin{aligned}
\dot{q} &= \sum_{m=1}^{N} \frac{\partial \hat{q}}{\partial q_m}(q_n, \mathbf{A}_k) \dot{q}_m + \sum_{j=1}^{K} \frac{\partial \hat{q}}{\partial \mathbf{A}_j}(q_n, \mathbf{A}_k) : \dot{\mathbf{A}}_j \\
&= \sum_{m=1}^{N} \frac{\partial \hat{q}}{\partial q_m}(q_n, \mathbf{A}_k) \hat{\xi}_m(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k) + \sum_{j=1}^{K} \frac{\partial \hat{q}}{\partial \mathbf{A}_j}(q_n, \mathbf{A}_k) : \hat{\mathcal{A}}_j(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k) \\
&\equiv \hat{\xi}(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k) = \tilde{\xi}\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right),
\end{aligned}
\tag{5.18}
$$

which is an evolution equation of the same general form as the evolution Eq. (5.8) for the scalar internal variables q_n. If the relation (5.17) can be inverted to give $q_1 = \hat{q}_1(q, q_2, \ldots, q_N, \mathbf{A}_k)$, then we can replace the internal variable q_1 with q, and the relation (5.17) may be regarded as a transformation from the original internal variable q_1 to the new internal variable q.

Several authors (Lubliner, 1973; Freed et al., 1991) have argued that such transformations should be allowed to depend on the current state of the material. For the theory considered here, this means that if

$$q = \hat{q}(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k) = \tilde{q}\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right) \tag{5.19}$$

can be solved for $q_1 = \hat{q}_1(\mathbf{E}_e, \theta, q, q_2, \ldots, q_N, \mathbf{A}_k)$, then q should also be regarded as an internal variable. As these authors have noted, such a generalization implies that $\dot{q}$ depends linearly on the rates of the external variables, which in our case are $\dot{\mathbf{E}}_e$ (or $\dot{\tilde{\mathbf{T}}}$) and $\dot{\theta}$. They conclude that the evolution Eqs. (5.8) for the original internal variables should also be extended to include dependence on these rates so that the general form of the evolution equations for the internal variables is invariant under transformations of the form (5.19). We draw a different conclusion, namely, that transformations of the internal variables of the form (5.19) are inconsistent with the property of instantaneous thermoelastic response. Indeed, the dependence of $\dot{q}$ on $\dot{\mathbf{E}}_e$ and $\dot{\theta}$ implies that $\dot{q}$ will suffer jump discontinuities for certain continuous, piecewise

smooth processes $(\mathbf{F}(t), \theta(t))$.[28] Hence, q can be an internal variable only if it is independent of $\mathbf{E}_e$ and θ, that is, if (5.19) reduces to (5.17).

The results in the previous two paragraphs generalize to transformations of tensor internal variables. Recall that the tensor internal variables $\mathbf{A}_k$ are invariant under superposed rigid motions. If

$$\mathbf{A} = \hat{\mathbf{A}}(q_n, \mathbf{A}_k) \quad \text{and} \quad \mathbf{A}_1 = \tilde{\mathbf{A}}(q_n, \mathbf{A}, \mathbf{A}_2, \ldots, \mathbf{A}_K), \tag{5.20}$$

then $\mathbf{A}$ is also invariant under superposed rigid motions and satisfies an evolution equation of the same form, (5.9), as the $\mathbf{A}_k$. Thus $\mathbf{A}$ may be regarded as an internal variable itself and can be used in place of $\mathbf{A}_1$. If dependence on $\mathbf{E}_e$ or θ were allowed in (5.20), then $\dot{\mathbf{A}}$ would not be continuous for all continuous processes $(\mathbf{F}(t), \theta(t))$, and so the property of instantaneous thermoelastic response would be violated.

For the remainder of this subsection we consider a continuous, piecewise smooth process $(\mathbf{F}(t), \theta(t))$. For the general class of materials with instantaneous thermoelastic response, the tensor internal variables $\mathbf{A}_k$ and their rates $\dot{\mathbf{A}}_k$ are continuous by assumption, while for the special class of materials considered in this section, the continuity of $\mathbf{A}_k$ and $\dot{\mathbf{A}}_k$ is a derived property. In Section 4.3, we remarked that the constitutive relations for ψ, e, and $\mathbf{T}$ could also have been expressed in terms of tensor internal variables $\mathbf{B}_k$ which transform like the Cauchy stress under superposed rigid motions, that is, $\mathbf{B}_k^* = \mathbf{Q}\mathbf{B}_k\mathbf{Q}^T$ if $\mathbf{x}^* = \mathbf{Q}\mathbf{x}$. However, continuity of $\dot{\mathbf{B}}_k$ is not to be expected.

For example, consider the case where $\mathbf{B}_k = \mathbf{R}_e\mathbf{A}_k\mathbf{R}_e^T$. Since $\mathbf{R}_e$ is also continuous, so is $\mathbf{B}_k$, but

$$\dot{\mathbf{B}}_k = \left(\mathbf{R}_e\mathbf{A}_k\mathbf{R}_e^T\right)^{\cdot} = \mathbf{R}_e\dot{\mathbf{A}}_k\mathbf{R}_e^T + \boldsymbol{\Omega}_e\mathbf{B}_k - \mathbf{B}_k\boldsymbol{\Omega}_e, \tag{5.21}$$

where the skew tensor $\boldsymbol{\Omega}_e$ is the elastic spin tensor,

$$\boldsymbol{\Omega}_e = \dot{\mathbf{R}}_e\mathbf{R}_e^T. \tag{5.22}$$

Since $\dot{\mathbf{R}}_e$ can suffer a jump discontinuity at an instant when $\dot{\mathbf{F}}$ has a jump discontinuity, it follows that $\boldsymbol{\Omega}_e$, and hence $\dot{\mathbf{B}}_k$, need not be continuous. On the other hand, the corotational rate

$$\overset{\triangledown}{\mathbf{B}}_k \equiv \dot{\mathbf{B}}_k + \mathbf{B}_k\boldsymbol{\Omega}_e - \boldsymbol{\Omega}_e\mathbf{B}_k = \mathbf{R}_e\dot{\mathbf{A}}_k\mathbf{R}_e^T = \mathbf{R}_e\left(\mathbf{R}_e^T\mathbf{B}_k\mathbf{R}_e\right)^{\cdot}\mathbf{R}_e^T \tag{5.23}$$

is continuous, since both $\dot{\mathbf{A}}_k$ and $\mathbf{R}_e$ are continuous. Since $\mathbf{R}_e^* = \mathbf{Q}\mathbf{R}_e$ and $\mathbf{A}_k^* = \mathbf{A}_k$, $(5.23)_2$ implies that $\overset{\triangledown}{\mathbf{B}}_k^* = \mathbf{Q}\overset{\triangledown}{\mathbf{B}}_k\mathbf{Q}^T$. On setting $\dot{\mathbf{A}}_j = \mathbf{R}_e^T\overset{\triangledown}{\mathbf{B}}_j\mathbf{R}_e$ and $\mathbf{A}_k = \mathbf{R}_e^T\mathbf{B}_k\mathbf{R}_e$ in (5.9), we see that properly invariant evolution equations for the $\mathbf{B}_k$ are

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[28] See the discussion of the possible dependence of ρ_R on $\mathbf{E}_e$ and θ in Section 4.7.

$$\overset{\triangledown}{\mathbf{B}}_j = \mathbf{R}_\mathrm{e} \hat{\mathcal{A}}_j \big(\mathbf{E}_\mathrm{e}, \theta, q_n, \mathbf{R}_\mathrm{e}^\mathrm{T} \mathbf{B}_k \mathbf{R}_\mathrm{e} \big) \mathbf{R}_\mathrm{e}^\mathrm{T}. \tag{5.24}$$

Similarly, if $\mathbf{B}_k = \mathbf{F}_\mathrm{e} \mathbf{A}_k \mathbf{F}_\mathrm{e}^\mathrm{T}$ then $\dot{\mathbf{B}}_k$ need not be continuous at an instant when $\dot{\mathbf{F}}$ has a jump discontinuity, but the rate

$$\overset{\triangle}{\mathbf{B}}_k \equiv \dot{\mathbf{B}}_k - \mathbf{B}_k \mathbf{L}_\mathrm{e}^\mathrm{T} - \mathbf{L}_\mathrm{e} \mathbf{B}_k = \mathbf{F}_\mathrm{e} \dot{\mathbf{A}}_k \mathbf{F}_\mathrm{e}^\mathrm{T} = \mathbf{F}_\mathrm{e} \big(\mathbf{F}_\mathrm{e}^{-1} \mathbf{B}_k \mathbf{F}_\mathrm{e}^{-\mathrm{T}} \big)^{\boldsymbol{\cdot}} \mathbf{F}_\mathrm{e}^\mathrm{T} \tag{5.25}$$

is continuous and transforms as $\overset{\triangle}{\mathbf{B}}_k^* = \mathbf{Q} \overset{\triangle}{\mathbf{B}}_k \mathbf{Q}^\mathrm{T}$. Properly invariant evolution equations for the $\mathbf{B}_k$ are

$$\overset{\triangle}{\mathbf{B}}_j = \mathbf{F}_\mathrm{e} \hat{\mathcal{A}}_j \big(\mathbf{E}_\mathrm{e}, \theta, q_n, \mathbf{F}_\mathrm{e}^{-1} \mathbf{B}_k \mathbf{F}_\mathrm{e}^{-\mathrm{T}} \big) \mathbf{F}_\mathrm{e}^\mathrm{T}. \tag{5.26}$$

From $(5.25)_2$ and $(5.23)_2$ we see that for small elastic shear strains,

$$\overset{\triangle}{\mathbf{B}}_k \approx \left(\frac{\rho_\mathrm{R}}{\rho} \right)^{2/3} \overset{\triangledown}{\mathbf{B}}_k. \tag{5.27}$$

Similar results hold if $\mathbf{B}_k = \mathbf{F}_\mathrm{e}^{-\mathrm{T}} \mathbf{A}_k \mathbf{F}_\mathrm{e}^{-1}$.

5.3. The yield surface and the structural surfaces

Basic to the idea of metal plasticity is that, for a given temperature and microstructure, there is a limit to the stress that the material may sustain without undergoing plastic deformation. This idea is usually expressed by assuming that if the stress lies within a certain (possibly evolving) surface in stress space, then the response is purely elastic. For the rate dependent theory considered here, plastic deformation occurs only if the stress lies outside the yield surface. The rate of plastic deformation vanishes if the stress lies on or inside the yield surface and would typically be expected to increase as the stress moves further outside the yield surface. In the classical rate independent theory, the stress never lies outside the yield surface, so plastic deformation can occur only when the stress lies continually on the yield surface, which must evolve in a compatible way. In the present formulation, the rate independent theory is expected to hold only in the limit of quasi-static deformations.

It should be emphasized that the theoretical framework developed up to this point is independent of the assumed existence of a yield surface. The results are applicable to viscoplasticity theories with no explicit yield surface as well as to theories with single or multiple yield surfaces. Examples of theories with no explicit yield surface can be found in Anand (1985), Bodner and Lindenfeld (1995), and Krempl and Gleason (1996), although the latter two papers employ a different kinematic framework than used here. In the remainder of this subsection we consider a single explicit yield surface as outlined above.

The yield surface may be described in terms of yield functions $\tilde{f}$ or $\hat{f}$ as the set of points satisfying[29]

$$\tilde{f}\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right) = \hat{f}(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k) = 0. \tag{5.28}$$

For fixed values of the temperature and the internal variables, (5.28) describes corresponding yield surfaces in stress space and elastic strain space. For fixed values of the internal variables, (5.28) describes corresponding yield surfaces in stress–temperature space and strain–temperature space. It is conventional to assume that the yield functions are positive in the region outside the yield surface and negative in the region enclosed by the yield surface. Plastic deformation is occurring iff the stress or elastic strain lies outside the yield surface, that is,

$$\dot{\mathbf{F}}_p \neq 0 \quad \text{iff} \quad \tilde{f}\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right) = \hat{f}(\mathbf{E}_e, \theta, q_n, \mathbf{A}_k) > 0. \tag{5.29}$$

The inequality (5.29) should not be regarded as a loading condition. If this inequality is satisfied then $\dot{\mathbf{F}}_p \neq 0$ regardless of whether or not the stress is increasing.

Typically, the yield surface evolves during plastic deformation due to changes in the internal variables. However, we do not assume that the yield condition (5.29) necessarily governs the onset of changes in the internal variables. In the present theory, there need not be an exact correlation between changes in internal variables and changes in plastic deformation. This feature is completely contrary to the definition, apparently first given by Rice (1971) and subsequently used by many authors since, that increments of plastic strain occur as a consequence of increments in the internal variables.[30] It must be emphasized again that in the present view, plastic strain may be calculated as a consequence of deformation, but it is never regarded as an internal variable itself in any fundamental way or even directly connected to internal variables as suggested by Rice's definition. Indeed, the plastic strain and the internal variables must be calculated independently as suggested by two limiting cases. First, note that the internal variables may all saturate at extreme deformations, but plastic straining continues, that is, when the creation and annihilation of dislocations balance. In this case, $\dot{q}_m = 0$ and $\dot{\mathbf{A}}_j = 0$, but $\dot{\mathbf{F}}_p \neq 0$. At the other extreme, the stress state may lie well within the yield surface so that no plastic deformation can occur, but thermal excursions may alter the internal variables. In

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[29] We could have started with the (apparently more general) assumptions that the yield function depends on the current state $(\mathbf{F}_e, \theta, q_n, \mathbf{A}_k)$ and is invariant under superposed rigid motions. Then the reduced forms in (5.28) follow by the arguments used in Section 4.3. Note that the various yield functions are not unique since multiplication by any positive function of the same variables results in a function with the same properties.

[30] In the present notation, that definition may be written for the case of scalar internal variables as $\dot{\mathbf{E}}_p \equiv \sum_{n=1}^{N}(\partial \mathbf{E}/\partial q_n)\dot{q}_n$, where $\mathbf{E} = \mathbf{E}\left(\tilde{\mathbf{T}}_0, \theta, q_n\right)$. Recall that $\mathbf{E}$ is the total strain tensor and $\tilde{\mathbf{T}}_0$ is the second Piola–Kirchhoff stress tensor relative to the initial configuration [see (3.2) and (3.3)].

this case, which may be called static recovery, $\dot{q}_m \neq 0$ or $\dot{A}_j \neq 0$ for some m or j, but $\dot{F}_p = 0$.[31] Furthermore, in the case where one of the internal variables is a work hardening parameter κ, it is not inconsistent to have $\dot{\kappa} < 0$ even when $\dot{F}_p \neq 0$. This situation may be called dynamic recovery. If the theory is to include saturation and recovery, there can be no general direct relationship between increments of internal variables and increments of plastic strain.[32]

Recall that the rate of change of the temperature is governed by the energy balance Eq. (4.84). It is common to assume that the inelastic heating term Δ in (4.84) is approximately proportional to the plastic stress power, say

$$\Delta = \beta \mathcal{P}_p, \tag{5.30}$$

where β is a constant which represents the fraction of the rate of plastic work converted to heating. In stating this approximation, most authors cite the experimental work of Taylor and Quinney (1934) who concluded that approximately 85–95% of the plastic work is converted to heat. Thus β is often taken to be a constant between 0.85 and 0.95, with $1 - \beta$ interpreted as the fraction of plastic work stored in the material, that is, the stored energy of cold work. It has been known in the metallurgical literature for a long time that β, as defined by (5.30), is not a material constant but in fact may vary enormously with temperature, strain, and plastic strain rate. Recently, there have been both theoretical and experimental studies that attempt to clarify and model this phenomenon; see Bodner and Lindenfeld (1995), Kamlah and Haupt (1998), Rosakis et al. (2000), and the references therein. Here we limit ourselves to a few general remarks about the variable β. If any one of the variables δ, Δ, $\mathcal{P}_p$, $\mathcal{P}_p^s$, or $\rho_R \mathcal{P}_p^s$ depends only on the state variables, then (as we observed in Section 5.1) all of them do, and hence, by (5.30), so does β. In particular, the third criterion for the plastic flow rule considered in Section 5.1, namely, that the plastic stress power due to dislocation slip depends only on the state variables, guarantees that β depends only on the state variables. In view of the expression (4.85)₁ for Δ, it would appear unlikely for β to be constant in general. However, in the case where all internal variables saturate, it is clear from (4.85)₁ that $\beta = 1$. On the other hand, if significant dynamic recovery occurs with rapid release of energy previously stored as cold work, it would appear that β could even be greater than one. Finally, note that a relation of the form (5.30) is generally not possible in the case of static recovery at zero pressure. Indeed, since $\dot{F}_p = 0$ and $p = 0$, both $\mathcal{P}_p^s$ and $\mathcal{P}_p^v$ are zero, by (4.92). Hence $\mathcal{P}_p = 0$, but by (4.58)₁ we see that the inelastic heating Δ need not be zero.

As remarked above, we do not assume any general direct relationship between increments of internal variables and increments of plastic strain. Likewise, the conditions

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[31] Since $\rho_R = \hat{\rho}_R(q_n, A_k)$, ρ_R may also be changing during such a process, so that $\dot{F}_p \neq 0$. Measurements of the increase in ρ_R during annealing have been made by Clarebrough et al. (1952, 1955, 1956, 1957, 1962).

[32] Examples of constitutive relations for modeling recovery and saturation can be found in Brown et al.(1989), Bodner and Lindenfeld (1995), Bammann et al. (1995), and Kamlah and Haupt (1998). Busso (1998) has developed an internal state variable model for dynamic recrystallization.

for the onset of the evolution of the internal variables need not be governed by the yield function. Instead, we assume the existence of scalar functions $\tilde{f}_m$ and $\tilde{A}_j$, possibly distinct from the yield function $\tilde{f}$, such that

$$\dot{q}_m \neq 0 \quad \text{iff} \quad \tilde{f}_m\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right) > 0, \tag{5.31}$$

$$\dot{\mathbf{A}}_j \neq \mathbf{0} \quad \text{iff} \quad \tilde{A}_j\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right) > 0. \tag{5.32}$$

Of course, the above conditions can also be expressed in terms of the elastic strain $\mathbf{E}_e$. For fixed values of the internal variables, the conditions $\tilde{f}_m\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right) = 0$ and $\tilde{A}_j\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right) = 0$ describe surfaces in stress-temperature space. Each surface encloses a region within which the corresponding internal variable cannot evolve further from its present state. These surfaces may be referred to as structural surfaces. Depending on the physical interpretation of the internal variables, these structural surfaces may coincide with each other or with the yield surface at some, none, or all points. For the discussion of the existence and uniqueness of solutions to the evolution equations at the end of Section 4.4, the conditions (5.31) and (5.32) and the yield condition (5.29) are regarded as being incorporated in the evolution equations. For example, the function $\tilde{\xi}_m$ in (5.8) must be zero in the region described by the inequality $\tilde{f}_m\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right) \leqslant 0$.

The elastic range of the material is the set of states for which $\dot{\mathbf{F}}_p = \mathbf{0}$, $\dot{q}_m = 0$, and $\dot{\mathbf{A}}_j = \mathbf{0}$ for all m and j. For fixed values of the internal variables, the elastic range in stress-temperature space is the intersection of the regions described by the inequalities $\tilde{f}\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right) \leqslant 0$, $\tilde{f}_m\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right) \leqslant 0$, and $\tilde{A}_j\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right) \leqslant 0$. Thus the elastic range is described by an inequality of the form $\tilde{l}\left(\tilde{\mathbf{T}}, \theta, q_n, \mathbf{A}_k\right) \leqslant 0$, where the elastic limit function $\tilde{l}$ is the pointwise maximum of the functions $\tilde{f}, \tilde{f}_m$, and $\tilde{A}_j$. We assume that $\tilde{l}\left(\mathbf{0}, \theta_0, q_n, \mathbf{A}_k\right) < 0$, which implies that the unstressed intermediate configuration is in the elastic range. It follows that the material may be unloaded from any state within the elastic range without undergoing plastic deformation or changes in the internal variables.

Observe that for tensor internal variables $\mathbf{B}_k$ that transform like the Cauchy stress under superposed rigid motions, the analog of the condition (5.32) involves a corotational rate (see Section 5.2) rather than the material time derivative. If $\mathbf{B}_k \neq \mathbf{0}$ and $\dot{\mathbf{F}}_e \neq \mathbf{0}$, then $\dot{\mathbf{B}}_k$ will generally be nonzero even in the elastic range. For example, if $\mathbf{B}_k = \mathbf{R}_e \mathbf{A}_k \mathbf{R}_e^{\mathrm{T}}$ then by (5.23) we see that $\dot{\mathbf{B}}_k = \mathbf{\Omega}_e \mathbf{B}_k - \mathbf{B}_k \mathbf{\Omega}_e$ in the elastic range, so that $\dot{\mathbf{B}}_k \neq \mathbf{0}$ if $\mathbf{B}_k$ does not commute with the elastic spin tensor $\mathbf{\Omega}_e$. In particular, a back stress tensor that transforms like the Cauchy stress will generally evolve even during elastic deformations.

6. Discussion

In Section 5.1, we argued that a reasonable restriction on the flow rule is that the plastic stress power $\mathcal{P}_p^s$ due to dislocation slip should depend only on the state

variables. We observed that this property holds if any one of the following conditions is satisfied (for condition 3 we must also assume an elastically isotropic material):

1. $\dot{F}_p F_p^{-1}$ depends only on the state variables.
2. $\tilde{D}_p$ depends only on the state variables.
3. $\mathrm{sym}(\dot{F}_p F_p^{-1})$ depends only on the state variables.

We also noted that condition 1 implies conditions 2 and 3. Since neither $\tilde{D}_p$ nor $\mathrm{sym}(\dot{F}_p F_p^{-1})$ suffices to determine $\dot{F}_p F_p^{-1}$, only condition 1 determines a complete flow rule once the dependence on the state variables is specified. If condition 3 is assumed, then the flow rule can be completed by specifying a constitutive relation for $\mathrm{skw}(\dot{F}_p F_p^{-1})$. Similarly, if condition 2 is assumed, then the flow rule can be completed by specifying a constitutive relation for $\mathrm{skw}(C_e \dot{F}_p F_p^{-1})$; indeed, since $\tilde{D}_p = \mathrm{sym}(C_e \dot{F}_p F_p^{-1})$, the two relations determine $C_e \dot{F}_p F_p^{-1}$, and hence $\dot{F}_p F_p^{-1}$. We may interpret $\mathrm{skw}(\dot{F}_p F_p^{-1})$ and $\mathrm{skw}(C_e \dot{F}_p F_p^{-1})$ as tensor measures of plastic spin. Note that in either case, $\mathcal{P}_p^s$ depends only on the state variables even if these plastic spin tensors depend on F_p as wells the state variables.

On the other hand, we have argued that "the rate of plastic deformation" should depend only on the current state, so that part of the determination of the flow rule involves the choice of a particular measure of plastic deformation rate. If this measure is decomposed into some preferred measures of plastic shearing and plastic spin, then it seems reasonable to require that each of these measures depend only on the state variables. For the two cases considered above, this would mean that both $\mathrm{sym}(\dot{F}_p F_p^{-1})$ and $\mathrm{skw}(\dot{F}_p F_p^{-1})$ depend only on the state variables or that both $\tilde{D}_p$ and $\mathrm{skw}(C_e \dot{F}_p F_p^{-1})$ depend only on the state variables. It should be clear from the above discussion that each set of conditions is in fact equivalent to condition 1. Thus if we wish to consider flow rules outside of the class defined by condition 1 but for which conditions 2 or 3 hold, then we need to consider different preferred measures of plastic spin. Of course, measures of plastic shearing other than $\tilde{D}_p$ and $\mathrm{sym}(\dot{F}_p F_p^{-1})$ might also be considered. These topics are pursued in a follow-up paper (Scheidler and Wright, 2001).

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Classes of flow rules for finite viscoplasticity

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Abstract

Classes of flow rules for finite viscoplasticity are defined by assuming that certain measures for plastic strain rate and plastic spin depend on the state variables but not on the plastic deformation. It is shown that three of these classes are mutually exclusive for finite elastic strains. For small elastic shear strains, two of the three classes are approximately equivalent. A number of exact and approximate kinematic relations between the various measures for plastic strain rate and plastic spin are derived. Some inconsistent flow rules encountered in the literature are also discussed. Throughout the paper, arbitrarily anisotropic materials are considered, and some of the simplifications resulting from the assumption of isotropy are noted. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Constitutive behaviour; Elastic–viscoplastic material; Finite strain; Anisotropic material

1. Introduction

In a recent paper, Scheidler and Wright (2001), began a review of the fundamental ideas and continuum framework that was deemed necessary for a thermodynamically consistent theory of thermoviscoplasticity with internal variables. In that paper, the fundamental observation from which all else follows is that even in a heavily deformed crystalline solid, the average spacing between dislocations is much larger than the typical dimensions of the basic lattice. As a consequence, stretching and distortion of the crystal lattice must be the origin of stress and the transmission of forces in thermoplastic, as well as thermoelastic solids. A natural extension of this fundamental idea is that plastic deformation occurs when the current stress exceeds the current capacity of the lattice. The lattice then becomes unstable so that existing dislocations begin to move and new ones may be generated. Furthermore, the rate of

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plastic deformation increases in some sense with the degree to which stress exceeds the basic stress carrying capacity of the lattice.

The paper referred to above dealt only with the basic kinematics and thermodynamics of finite plastic deformation, relying on the standard multiplicative decomposition of the deformation gradient into elastic and plastic parts. This paper takes up where the previous one left off, focusing on the notion of the evolutionary rate of plastic slip, that is, the plastic flow rule. In particular, we examine the consequences of demanding that all constitutive quantities, including the evolutionary rates of plastic deformation and internal variables, must depend only on internal and thermoelastic state variables, but in no way upon plastic slip itself. The reason for this restriction is that many paths of plastic deformation may lead to the same thermoelastic and internal state, as expressed by stress, temperature, and the set of internal variables. Thus, the plastic part of the deformation, in particular the slip, cannot itself be part of the prescription of internal state.

Although the basic motivation in the last paragraph is clear, its mathematical expression is not entirely straightforward. The reason for this is that there is no unique way to choose a kinematic term to measure "the rate of plastic slip," and in fact several different measures have been chosen in the literature. While it is often possible to express one measure for the rate of plastic slip in terms of another, such expressions often involve an explicit dependence on the plastic slip itself. When that is the case and one measure for the rate of plastic slip depends only on the internal and thermoelastic state variables, then the other measure must necessarily depend explicitly on the plastic slip as well as the state variables. With the restriction that plastic slip is not allowed to be an internal variable, it follows that the two different measures for the rate of plastic slip are not equivalent in the sense that both measures cannot depend on the state variables alone.

It is clear that determination of the flow rule, which is one of the central problems for plasticity theory, actually consists of two parts. First, there is the choice of a preferred measure for the rate of plastic slip—"preferred" in the sense that this measure is assumed to depend on the state variables only. Second, there is the choice of the particular flow function that relates the preferred measure to the state variables, but in this paper we do not pursue this aspect of the flow rule further. Instead, we regard the flow function as essentially arbitrary, so that each preferred measure for the rate of plastic slip defines an entire class of flow rules. One of the main problems addressed in this paper is whether two classes defined by different measures for the rate of plastic slip have any flow rules in common.

The paper proceeds as follows. Section 2 begins with a brief reiteration of notation and fundamental kinematics. This is followed by properly invariant statements of the thermoelastic constitutive relations and the evolution equations for the internal state variables. The internal and free energies, the second Piola–Kirchhoff stress, and the evolutionary rates of the internal variables are assumed to depend on the state variables but not on the plastic slip (i.e., the isochoric part of the plastic deformation), although they may depend on the volumetric part of the plastic deformation since the intermediate density may itself be an internal variable. All constitutive relations are formulated in the intermediate configuration.

Three restrictions on the viscoplastic flow rule are imposed in Section 3. The first has already been discussed above, namely, that some preferred measure for the rate of plastic slip should depend on the state variables but not on the plastic slip itself. The second restriction requires that the stress power due to plastic slip should also depend only on the state variables. The third restriction requires that, regardless of the preferred measure for the rate of plastic slip, the plastic velocity gradient depends at most on the state variables and the plastic slip. The second and third restrictions limit the choices of preferred measures for the rate of plastic slip. The possibility that the plastic velocity gradient depends on the plastic slip cannot be ruled out if certain measures for the rate of plastic slip other than the plastic velocity gradient are to depend only on the state variables.

In continuum models for polycrystals the flow rule often takes the form of separate constitutive relations for a "plastic strain rate" and a "plastic spin." We follow this practice in most of the paper. Consequently, the first restriction mentioned above is relaxed by requiring that some preferred measures for plastic strain rate and plastic spin depend only on the state variables. Several choices have been made by various authors in the plasticity literature for the proper measures of plastic strain rate and plastic spin. A few examples are discussed in Section 3, and in Section 4 we compare six representative choices. It turns out that four of these six choices define equivalent classes of flow rules, so that there are actually only three mutually exclusive classes. No material symmetry restrictions are imposed, although one of the classes generally fails to satisfy the criterion for the plastic stress power unless the material is elastically isotropic. Section 4 closes with a discussion of the special nature of isotropic materials with only scalar internal variables.

Results stated without proof in Section 4 are derived in Sections 5 and 6. The proofs rely on a number of purely kinematic relations between the various measures for plastic strain rate and plastic spin. In Section 5 we complete the proof that four of the six classes of flow rules introduced in Section 4 are equivalent. The proof that this class and the remaining two classes are mutually exclusive is given in Section 6. Both sections contain further discussion of the role of plastic spin.

The results up to this point in the paper are valid without restrictions on the magnitude of the elastic part of the deformation. The approximations following from the assumption that elastic shear strains are "small" are carefully examined in Section 7. A subtle point that seems to have been overlooked by some authors is that in the approximation of one measure for plastic strain rate in terms of another, the products of elastic shear strain and plastic spin terms are not necessarily negligible unless the plastic spin is bounded by the plastic strain rate. Several equivalent statements of this condition, involving different plastic strain rate and plastic spin measures, are considered, and it is shown that whenever such bounds hold, two of the mutually exclusive classes of flow rules considered in Section 4 are approximately equivalent for small elastic shear strains.

For some flow rules encountered in the plasticity literature, the chosen measure for the rate of plastic slip is inconsistent with the variables on which it is assumed to depend. Section 8 contains a discussion of this issue, based in part on observations by Nemat-Nasser (1990, 1992).

In Section 9 we examine a fourth class of flow rules which is motivated by the results in Section 8. It is shown that flow rules in this fourth class that do not belong to one of the classes considered previously fail to satisfy the criterion for the plastic stress power.

The paper closes with a discussion of the main results in Section 10.

Appendix A contains some purely algebraic results used in the derivation the kinematic relations for plastic strain rate and plastic spin. Some of the more technical details in the derivation of these kinematic relations have been relegated to Appendix B.

2. Preliminaries

The kinematics of finite elastic–plastic deformation is based on the multiplicative decomposition of the total deformation gradient F into an elastic part F_e and a plastic part F_p,

$$F = F_e F_p. \tag{2.1}$$

It is conceptually useful to interpret F_p as mapping a neighborhood of a point in the undistorted initial configuration onto a local, plastically deformed, intermediate configuration at the initial temperature. Then F_e is interpreted as a thermoelastic deformation of this intermediate configuration onto a neighborhood of a point in the current configuration at the current absolute temperature θ.

The plastic deformation F_p can be decomposed into an isochoric part $\bar{F}_p$ representing plastic slip and a dilatational part, $(\det F_p)^{1/3}$, representing any plastic volume change due to changing numbers of dislocations[1] or the nucleation and growth of voids. If ρ_0 and ρ_R denote the densities in the initial and intermediate configurations, respectively, then $\det F_p = \rho_0/\rho_R$. In Scheidler and Wright (2001), Section 4.7, it was shown that the entropy inequality and the property of instantaneous thermoelastic response imply that the density ρ_R in the intermediate configuration can be at most a function of the internal state variables. The scalar internal variables are denoted by $q_1, \ldots, q_N$ and the tensor internal variables by $A_1, \ldots, A_K$. Depending on the context, we use q_n and A_k to denote either typical scalar and tensor internal variables or the lists of these variables. Then we have

$$F_p = \left(\frac{\rho_0}{\rho_R}\right)^{1/3} \bar{F}_p, \qquad \det \bar{F}_p = 1, \qquad \rho_R = \hat{\rho}_R(q_n, A_k). \tag{2.2}$$

The relation (2.2)$_3$ includes the special case where ρ_R is itself an internal variable, say $\rho_R = q_1$. Plastic incompressibility, which is not assumed here, can be imposed by taking $\rho_R \equiv \rho_0$, in which case $F_p = \bar{F}_p$.

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[1] References to the literature on this subject are given in Scheidler and Wright (2001), Section 2. See also the recent paper by Altenbach et al. (2001) for a discussion of plastic volume change in grey cast iron.

The tensors F_e and F_p have unique polar decompositions

$$F_e = R_e U_e = V_e R_e, \qquad F_p = R_p U_p = V_p R_p. \tag{2.3}$$

R_e and R_p are the local elastic and plastic rotation tensors, respectively. The symmetric positive-definite tensors U_e and V_e (U_p and V_p) are the right and left elastic (plastic) stretch tensors. If the density in the current configuration is denoted by ρ, then

$$\det F_e = \det U_e = \det V_e = \frac{\rho_R}{\rho}, \qquad \det U_p = \det V_p = 1. \tag{2.4}$$

The elastic finite strain tensor E_e is defined by

$$E_e = \frac{1}{2}(C_e - I), \qquad C_e = F_e^T F_e = U_e^2. \tag{2.5}$$

The Cauchy stress tensor is denoted by T. The second Piola–Kirchhoff stress tensor $\tilde{T}$ relative to the intermediate configuration is given by

$$\tilde{T} = (\det F_e) F_e^{-1} T F_e^{-T} = \frac{\rho_R}{\rho} F_e^{-1} T F_e^{-T}. \tag{2.6}$$

We assume that F_p and the internal state variables q_n and A_k are invariant under superposed rigid motions. It follows that F_p, R_p, U_p, and V_p as well as U_e, C_e, E_e, and $\tilde{T}$ are also invariant. For reasons discussed in the Introduction [see also Scheidler and Wright (2001), Section 5.1], F_p is not regarded as an internal state variable. Similarly, any functions of F_p, such as R_p, U_p, and V_p, are excluded from the list of internal variables. The state of the material is characterized by the thermomechanical variables F_e and θ and the internal variables q_n and A_k. For the constitutive relations considered in this paper, which may be thought of as formulated in the intermediate configuration, any dependence on F_e arises only through U_e or, equivalently, E_e. It will be convenient to let S denote any list of (reduced) state variables, for example,

$$S = (U_e, \theta, q_n, A_k) \quad \text{or} \quad (E_e, \theta, q_n, A_k). \tag{2.7}$$

Of course, an equivalent list can be obtained by replacing U_e with any invertible function of U_e, such as $U_e^2 = C_e$. Also, the entropy η may be used instead of θ as the independent thermodynamic variable.

Let e and $\psi = e - \theta\eta$ denote the internal energy and free energy, respectively. We restrict attention to materials that satisfy thermoelastic constitutive relations of the form

$$e = \tilde{e}(S), \qquad \psi = \tilde{\psi}(S), \qquad \tilde{T} = \tilde{T}(S), \tag{2.8}$$

and evolution equations for the internal variables of the form

$$\dot{q}_m = \xi_m(\mathcal{S}), \qquad \dot{A}_j = \mathcal{A}_j(\mathcal{S}), \qquad\qquad (2.9)$$

where a superposed dot denotes the material time derivative. In other words, the internal and free energies, the second Piola–Kirchhoff stress, and the evolutionary rates of the internal variables depend only on the state variables. Motivation for these assumptions has been given in Scheidler and Wright (2001), Section 5. We also assume that the stress–strain relation (2.8)₃ is invertible in the elastic strain E_e, so tha $E_e = \mathcal{E}(\tilde{T}, \theta, q_n, A_k)$. Then the second Piola–Kirchhoff stress tensor $\tilde{T}$ may be regarded as a state variable, and we may take

$$\mathcal{S} = \left(\tilde{T}, \theta, q_n, A_k \right) \qquad\qquad (2.10)$$

in the above relations.

The constitutive relations (2.8), (2.9), and (2.2) are properly invariant. Of course, the constitutive functions in (2.8) and (2.9) depend on the particular list of state variables represented by $\mathcal{S}$.

3. Restrictions on the flow rule

To complete the constitutive framework outlined in the previous section we must specify the general form of the viscoplastic flow rule, that is, the form of the evolution equation for the plastic slip F_p. The second law of thermodynamics (in the form of the Clausius–Duhem inequality) requires that the flow rule be consistent with the plastic dissipation inequality, which states that the rate of change of the stored energy of cold work cannot exceed the rate of plastic work (Scheidler and Wright, 2001, Sections 4.6 and 4.8). This imposes only mild restrictions on the flow rule. Consequently, a number of authors have postulated additional thermodynamic criteria from which more severe restrictions on the flow rule can be derived. An example of such a criterion is a "maximum dissipation principle"; see Rajagopal and Srinivasa (1998), Mollica et al. (2001), Srinivisa (2001), Lubliner (1984), Deseri and Mares (2000), Cermelli et al. (2001), and the references cited therein for various formulations and implications of this principle.

3.1. General restrictions on the flow rule

In this paper we consider some relatively weak restrictions on the flow rule. In particular, we examine the consequences of demanding that the flow rule satisfies the following criteria (Scheidler and Wright, 2001, Sections 4.4 and 5.1):

1. Some preferred measure for the (total) rate of plastic slip depends only on the current state of the material and thus is a function of the state variables S only. Alternatively, some preferred measures for plastic strain rate and plastic spin depend only on the state variables S.[2]

2. The stress power due to plastic slip also depends only on the state variables S.

3. $\dot{F}_p$ depends at most on the state variables S and the current value of F_p.

A fourth criterion involving bounds on the plastic spin is considered in Section 7 in relation to the approximations for small elastic shear strains; see also the discussion in Section 10. These restrictions do not seem to follow from any maximum dissipation principle. On the other hand, for rate-dependent materials they are not necessarily inconsistent with a maximum dissipation postulate and thus might be invoked in conjunction with it.

The third criterion above ensures that the constitutive theory is complete in the sense that once all constitutive functions are specified, the values of the internal variables and the plastic deformation F_p as well as the values of the thermoelastic variables F_e, $\tilde{T}$, T, ψ, θ, and η are determined at any time t by the history of F and θ up to time t and the initial values of the internal variables.[3] Furthermore, the materials described by these relations have the property of instantaneous thermoelastic response (Scheidler and Wright, 2001, Section 4.4). This property ensures that incremental loading waves in a plastically prestressed material travel at the elastic wave speed and that strain rate jump tests show an initially elastic transient as the flow stress switches continuously from one constant strain rate to another. These properties would not hold if any of the rates $\dot{\tilde{T}}$, $\dot{F}$, $\dot{E}_e$, or $\dot{\theta}$ were included in the list of arguments in the evolution equations for F_p or the internal state variables.

Motivation for the first criterion above was given in the Introduction. The second criterion is more difficult to motivate physically. As discussed below, all three criteria are satisfied by viscoplastic slip models for single crystals as developed, for example, in Teodosiu and Sidoroff (1976). The fact that the second criterion is satisfied by these physically based theories was the primary motivation for including it here. We do not claim that models for which this plastic stress power criterion does not hold are necessarily unphysical, only that this criterion seems worthy of consideration.

Regarding the second criterion, we recall that the plastic stress power per unit mass, $\mathcal{P}_p$, can be decomposed into a contribution $\mathcal{P}_p^s$ from plastic slip and a contribution $\mathcal{P}_p^v$ from plastic volume change, and that given the assumptions in Section 2, $\mathcal{P}_p^v$ necessarily depends only on the state variables (Scheidler and Wright, 2001, Sections 4.8 and 5.1). Hence, the second criterion above, namely, that $\mathcal{P}_p^s$ depends only on the state variables, is equivalent to the requirement that the (total) plastic stress power $\mathcal{P}_p$ depends only on the state variables. Also note that $\rho_R \mathcal{P}_p^s$ is the stress power due to plastic slip, measured per unit volume in the intermediate configuration.

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[2] As discussed below, this alternative criterion is less restrictive than the analogous criterion for the total rate of plastic slip.

[3] It is assumed that $F_p = I$, and hence $\dot{F}_p = I$, initially.

In view of the fact that ρ_R is a function of the internal variables or is itself an internal variable, the requirement that $\mathcal{P}_p^s$ depends only on the state variables is equivalent to the requirement that $\rho_R \mathcal{P}_p^s$ depends only on the state variables. Since the contribution $\mathcal{P}_p^V$ from the rate of plastic volume change is not discussed further in this paper, we will refer to $\mathcal{P}_p^s$ or $\rho_R \mathcal{P}_p^s$ simply as the "plastic stress power." A general expression for the plastic stress power is

$$\rho_R \mathcal{P}_p^s = C_e \tilde{T} : \dot{F}_p F_p^{-1} \tag{3.1}$$

(Scheidler and Wright, 2001, Section 4.8).

Although we do not invoke a maximum dissipation principle here, our second criterion does place some restrictions on the dissipation. Indeed, the constitutive assumptions in Section 2, the third criterion above, and the Clausius–Duhem inequality imply that the following conditions are equivalent:

2. The plastic stress power depends only on the state variables.

2.′ The (rate of) internal dissipation δ depends only on the state variables.

2.″ The inelastic heating Δ depends only on the state variables.

See Scheidler and Wright (2001), Section 5.1.

An equivalent statement of the third criterion is that the "plastic velocity gradient" $\dot{F}_p F_p^{-1}$ must ultimately be expressible in the form

$$\dot{F}_p F_p^{-1} = \Phi_p(\mathcal{S}, F_p), \tag{3.2}$$

regardless of the preferred measures alluded to in the first criterion. Since $\text{tr}\left(\dot{F}_p F_p^{-1}\right) = 0$, the function Φ_p in (3.2) must be deviatoric. Note that any flow rule of the form (3.2) is properly invariant. Now suppose that the preferred measure for the rate of plastic slip is taken to be the plastic velocity gradient. In view of the assumption that F_p is not a state variable, both the first and the third criteria above can be satisfied by eliminating the dependence of the function Φ_p on F_p in (3.2). The class of flow rules of this form is designated Class I.

Class I: $\dot{F}_p F_p^{-1} = \Phi_p(\mathcal{S}).$ \hfill (3.3)

Since C_e and $\tilde{T}$ are state variables, the constitutive relation (3.3) and the expression (3.1) for the plastic stress power imply that the plastic stress power depends only on the state variables $\mathcal{S}$. Thus the second criterion above is also satisfied for flow rules in Class I. Flow rules in this class arise in viscoplastic slip models for a single crystal or a polycrystalline grain (Teodosiu and Sidoroff, 1976).

While flow rules in Class I are also used in continuum models for polycrystals, in view of the complicating effects of grain boundaries it is not clear that a mathematical description of viscoplastic flow in polycrystals should be limited to flow rules in this

class. We also wish to consider flow rules outside Class I but still satisfying the three criteria discussed above. Regarding the third criterion [in the form (3.2)], the possibility that the plastic velocity gradient may depend on the plastic slip F_p cannot be excluded if certain measures for the rate of plastic slip other than the plastic velocity gradient are to depend only on the state variables.

3.2. Preferred measures for plastic strain rate and plastic spin: general remarks

In the continuum modeling of polycrystals it is common practice to decompose a preferred measure for the total rate of plastic slip, e.g., $\dot{F}_p F_p^{-1}$, into a plastic strain rate and plastic spin and then specify separate constitutive relations for the latter two measures. These two constitutive relations form the flow rule. More generally, even if a preferred measure for the total rate of plastic slip does not suggest itself, theory or experiment may suggest preferred measures for plastic strain rate and plastic spin—preferred in the sense that the current values of these measures are expected to depend only on the current state of the material. If $\mathbb{D}_p$ denotes the preferred measure for plastic strain rate[4] and $\mathbb{W}_p$ the preferred measure for plastic spin, then the flow rule takes the form

$$\mathbb{D}_p = \mathcal{D}_p(\mathcal{S}), \qquad \mathbb{W}_p = \mathcal{W}_p(\mathcal{S}). \tag{3.4}$$

Since the list $\mathcal{S}$ of state variables is invariant under superposed rigid motions, $\mathbb{D}_p$ and $\mathbb{W}_p$ must also have this property and thus may be thought of as residing in the intermediate configuration. Of course, (3.4) may always be transformed to an equivalent flow rule in the current configuration, but such transformations will not be discussed here.

We emphasize that (3.4) is regarded as a flow rule, not as a definition of the preferred measures $\mathbb{D}_p$ and $\mathbb{W}_p$. $\mathcal{D}_p$ and $\mathcal{W}_p$ are constitutive functions of the state variables only and thus do not depend on F_p itself, whereas the definitions of the symmetric tensor $\mathbb{D}_p$ and the skew tensor $\mathbb{W}_p$ are purely kinematic and may be assumed to have the general form

$$\mathbb{D}_p = \hat{\mathbb{D}}_p\left(\dot{F}_p, F_p, C_e\right), \qquad \mathbb{W}_p = \hat{\mathbb{W}}_p\left(\dot{F}_p, F_p, C_e\right). \tag{3.5}$$

Motivation for including a possible dependence on C_e (equivalently, U_e or E_e) in (3.5) is given by the example at the end of this section. Of course, not all relations of the form (3.5) lead to physically reasonable measures for plastic strain rate and plastic spin, but it is not our intention to address this issue here. Instead, we will be content to study a few representative examples. Nevertheless, it is clear that if the

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[4] The total plastic strain rate contains contributions from plastic volume change as well as plastic slip. To distinguish between the two, the latter contribution was referred to as plastic shearing in Scheidler and Wright (2001). Since the rate of plastic volume change is not discussed further in this paper, we use the term "plastic strain rate" to refer to the contribution from plastic slip only.

third criterion above is to be satisfied for a flow rule of the form (3.4), that is, if the flow must ultimately be expressible in the form (3.2), then $\mathbb{D}_p$ and $\mathbb{W}_p$ must essentially determine $\dot{F}_p F_p^{-1}$. In this regard, note that if there exists a *kinematic* relation of the form

$$\dot{F}_p F_p^{-1} = v(\mathbb{D}_p, \mathbb{W}_p, F_p, C_e), \tag{3.6}$$

then substitution of the flow rule (3.4) into this relation yields an alternate expression for the flow rule in terms of the plastic velocity gradient:

$$\dot{F}_p F_p^{-1} = v(\mathcal{D}_p(\mathcal{S}), \mathcal{W}_p(\mathcal{S}), F_p, C_e). \tag{3.7}$$

And since C_e is a state variable, the third criterion (3.2) will be satisfied. Therefore we wish to confine our attention to plastic strain rate and plastic spin measures $\mathbb{D}_p$ and $\mathbb{W}_p$ for which a kinematic relation of the form (3.6) holds. For some of the measures considered in this paper, it is a nontrivial task to verify that a relation of the form (3.6) follows from the given definitions of $\mathbb{D}_p$ and $\mathbb{W}_p$ [in the form of special cases of (3.5)].

A complete description of the flow rule requires specification of the preferred plastic strain rate and spin measures [i.e., the functions $\hat{\mathbb{D}}_p$ and $\hat{\mathbb{W}}_p$ in (3.5)] and of the flow functions $\mathcal{D}_p$ and $\mathcal{W}_p$ in (3.4). The latter problem is outside the scope of this paper. Instead, we regard $\mathcal{D}_p$ and $\mathcal{W}_p$ as arbitrary symmetric and skew tensor-valued functions of the state variables,[5] so that each choice of $\mathbb{D}_p$ and $\mathbb{W}_p$ defines an entire class of flow rules. These classes include the special case where the spin function $\mathcal{W}_p$ is identically zero, that is, flow rules for which the preferred plastic spin measure $\mathbb{W}_p \equiv 0$ for all motions.

3.3. Preferred measures for plastic strain rate and plastic spin: some examples

As a simple example, let the preferred plastic strain rate and plastic spin be the symmetric and skew parts of the plastic velocity gradient, that is,

$$\mathbb{D}_p = \text{sym}\left(\dot{F}_p F_p^{-1}\right), \qquad \mathbb{W}_p = \text{skw}\left(\dot{F}_p F_p^{-1}\right). \tag{3.8}$$

Then the assumption that these measures depend only on the state variables results in a class of flow rules, designated Class I.a, of the general form

$$\textbf{Class I.a:} \qquad \text{sym}\left(\dot{F}_p F_p^{-1}\right) = \mathcal{D}_p(\mathcal{S}), \qquad \text{skw}\left(\dot{F}_p F_p^{-1}\right) = \mathcal{W}_p(\mathcal{S}). \tag{3.9}$$

Clearly, this class of flow rules is equivalent to Class I introduced above, with $\Phi_p = \mathcal{D}_p + \mathcal{W}_p$, $\mathcal{D}_p = \text{sym}\Phi_p$, and $\mathcal{W}_p = \text{skw}\Phi_p$. In particular, for flow rules in

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[5] Actually, some restrictions may need to be imposed on these functions, depending on the choice of $\mathbb{D}_p$ and $\mathbb{W}_p$. For example, if $\mathbb{D}_p$ is taken to be $\text{sym}\left(\dot{F}_p F_p^{-1}\right)$, which is deviatoric, then we must have $\text{tr}\mathcal{D}_p = 0$.

Class I.a the plastic stress power depends only on the state variables. Since the constitutive function $\mathcal{W}_p \equiv 0$ is not excluded, Class I.a includes flow rules for which $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) \equiv 0$ for all motions.

Note that for flow rules in Class I.a, the partitioning of the total rate of plastic slip (as measured by the plastic velocity gradient) into a plastic strain rate and a plastic spin is not absolutely necessary, since we can also express the plastic velocity gradient directly as a function of the state variables, as in (3.3). On the other hand, suppose we choose some measure $\mathcal{W}_p$ for the preferred plastic spin other than $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$, but continue to take $\mathbb{D}_p = \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$. Then it may not be obvious how to combine $\mathbb{D}_p$ and $\mathcal{W}_p$ to form a tensor that measures the total rate of plastic slip; for example, simply adding $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\mathcal{W}_p$ may not make sense kinematically. Thus the requirement that some measures for plastic strain rate and plastic spin depend only on the state variables would seem to be less restrictive than requiring that some measure for the total rate of plastic slip depends only on the state variables. Of course, if a kinematic relation of the form (3.6) exists, then $\mathbb{D}_p$ and $\mathcal{W}_p$ do combine in some way to form the plastic velocity gradient, but if the dependence of $\dot{F}_p F_p^{-1}$ on F_p in the kinematic relation (3.6) is nontrivial, then the dependence of $\dot{F}_p F_p^{-1}$ on F_p will also be nontrivial in the flow rule (3.7), in which case $\dot{F}_p F_p^{-1}$ does not depend on the state variables alone for this class of flow rules.

When $\mathbb{D}_p = \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\mathcal{W}_p$ is a measure for plastic spin other than $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$, it is also necessary to check that the plastic stress power depends only on the state variables. In this regard, observe that if C_e and $\tilde{T}$ commute, then $C_e \tilde{T}$ is a symmetric tensor, so in the inner product (3.1) we may replace $\dot{F}_p F_p^{-1}$ by its symmetric part:

$$\rho_R \mathcal{P}_p^s = C_e \tilde{T} : \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right), \qquad \text{if} \qquad C_e \tilde{T} = \tilde{T} C_e. \tag{3.10}$$

Although C_e and $\tilde{T}$ need not commute in general, they do commute if the material is elastically isotropic, that is, if

$$\tilde{T} = \tilde{\mathcal{T}}(E_e, \theta, q_n) \tag{3.11}$$

for some isotropic function $\tilde{\mathcal{T}}$.[6] Since C_e and $\tilde{T}$ are state variables, if we take $\mathbb{D}_p = \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ then the expression (3.10) for the plastic stress power and the constitutive assumption $\mathbb{D}_p = \mathcal{D}_p(\mathcal{S})$ imply that for an elastically isotropic material, the plastic stress power depends only on the state variables $\mathcal{S}$, regardless of the choice for the plastic spin tensor $\mathcal{W}_p$.

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[6] It is easily shown that an equivalent relation in terms of the Cauchy stress is $T = \hat{\mathcal{T}}(V_e, \theta, q_n)$ for some isotropic function $\hat{\mathcal{T}}$. This in turn is equivalent to the condition that $T = \check{\mathcal{T}}(F_e, \theta, q_n) = \check{\mathcal{T}}(F_e H, \theta, q_n)$ for any rotation H.

We close this section by motivating a different choice for $\mathbb{D}_p$. Recall that another general expression for the plastic stress power is (Scheidler and Wright, 2001, Section 4.8)

$$\rho_R \mathcal{P}_p^s = \tilde{T} : \tilde{D}_p,$$ (3.12)

where the plastic strain rate tensor $\tilde{D}_p$ is defined by[7]

$$\tilde{D}_p = \text{sym}\left(C_e \dot{F}_p F_p^{-1}\right).$$ (3.13)

Since $\tilde{T}$ is a state variable, if we take $\mathbb{D}_p = \tilde{D}_p$ then the relation (3.12) for the plastic stress power and the constitutive assumption $\mathbb{D}_p = \mathcal{D}_p(\mathcal{S})$ imply that the plastic stress power depends only on the state variables $\mathcal{S}$, regardless of the choice of the plastic spin tensor $\mathbb{W}_p$ in (3.4). Thus any flow rule of the form (3.4) with $\mathbb{D}_p = \tilde{D}_p$ satisfies our first two criteria. However, the choice of the plastic spin tensor $\mathbb{W}_p$ is restricted by the third criterion, that is, $\mathbb{W}_p$ must be such that a relation of the form (3.6) holds with $\mathbb{D}_p = \tilde{D}_p$.

4. Classes I–III: summary of results

In this section we consider six classes of flow rules of the form (3.4)—the different classes being defined by different choices for $\mathbb{D}_p$ and $\mathbb{W}_p$. For the preferred plastic strain rate $\mathbb{D}_p$ we consider the tensors $\text{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\tilde{D}_p$ discussed in Section 3. For the preferred plastic spin $\mathbb{W}_p$ we consider the tensors $\text{skw}\left(\dot{F}_p F_p^{-1}\right)$, $\tilde{W}_p$, and Ω_p. The plastic spin tensor $\tilde{W}_p$ is defined by analogy with $\tilde{D}_p$:

$$\tilde{W}_p = \text{skw}\left(C_e \dot{F}_p F_p^{-1}\right).$$ (4.1)

The skew tensor Ω_p is the commonly used measure of plastic spin defined by[8]

$$\Omega_p = \dot{R}_p R_p^T.$$

The six classes considered here are formed by taking all combinations of these three measures of plastic spin and the two measures of plastic strain rate. These classes of

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[7] This measure of plastic strain rate has been considered by several other authors; see Moran et al.(1990), Armero and Simo (1993), Miehe (1994), Maugin (1994), Maugin and Epstein (1998), and Cleja-Ţigoiu and Maugin (2000).

[8] Note that since $R_p = F_p U_p^{-1}$ and $U_p = \sqrt{F_p^T F_p}$, $\Omega_p = \hat{W}_p\left(\dot{F}_p, F_p\right)$ for some function $\hat{W}_p$, so that (3.5)$_2$ is satisfied for $\mathbb{W}_p = \Omega_p$.

Table 1
Classes of flow rules of the form $\mathbb{D}_p = \mathcal{D}_p(\mathcal{S})$ and $\mathbb{W}_p = \mathcal{W}_p(\mathcal{S})$, where $\mathcal{S}$ denotes a list of state variables

Class	$\mathbb{D}_p$	$\mathbb{W}_p$	Derived properties
I.a	$\mathrm{sym}\!\left(\dot{F}_p F_p^{-1}\right)$	$\mathrm{skw}\!\left(\dot{F}_p F_p^{-1}\right)$	Classes I.a–I.d are equivalent to Class I.
I.b	$\tilde{D}_p$	$\tilde{W}_p$	Ω_p depends on V_p and $\mathcal{S}$.
I.c	$\mathrm{sym}\!\left(\dot{F}_p F_p^{-1}\right)$	$\tilde{W}_p$	$\mathrm{skw}\!\left(\dot{F}_p F_p^{-1}\right) = 0 \not\Leftrightarrow \tilde{W}_p = 0$, unless C_e and $\mathrm{sym}\!\left(\dot{F}_p F_p^{-1}\right)$ commute.
I.d	$\tilde{D}_p$	$\mathrm{skw}\!\left(\dot{F}_p F_p^{-1}\right)$	The plastic stress power depends only on $\mathcal{S}$.
II	$\mathrm{sym}\!\left(\dot{F}_p F_p^{-1}\right)$	Ω_p	$\tilde{D}_p$, $\mathrm{skw}\!\left(\dot{F}_p F_p^{-1}\right)$, and $\tilde{W}_p$ depend on V_p and $\mathcal{S}$. The plastic stress power depends on V_p and $\mathcal{S}$, unless the material is elastically isotropic.
III	$\tilde{D}_p$	Ω_p	$\mathrm{sym}\!\left(\dot{F}_p F_p^{-1}\right)$, $\mathrm{skw}\!\left(\dot{F}_p F_p^{-1}\right)$, and $\tilde{W}_p$ depend on V_p and $\mathcal{S}$. The plastic stress power depends only on $\mathcal{S}$.

flow rules are listed in Table 1 along with some of their properties. The results in Table 1 and elsewhere in this and the next two sections are valid for finite elastic strains. Approximations for small elastic shear strains are discussed in Section 7. Another class of flow rules is studied in Section 8.

Since $\mathrm{sym}\!\left(\dot{F}_p F_p^{-1}\right)$ is deviatoric, for Classes I.a, I.c, and II the flow function $\mathcal{D}_p$ must satisfy $\mathrm{tr}\,\mathcal{D}_p = 0$. Since $\mathrm{tr}\!\left(C_e^{-1}\tilde{D}_p\right) = 0$, for Classes I.b, I.d, and III the flow function $\mathcal{D}_p$ must satisfy $\mathrm{tr}\!\left(C_e^{-1}\mathcal{D}_p(\mathcal{S})\right) = 0$.

4.1. Equivalent classes of flow rules

Two classes of flow rules are said to be *equivalent* if every flow rule in one class can be expressed as a flow rule in the other class and vice versa. In Section 3 we observed that Class I.a is equivalent to Class I, as defined by (3.3). As indicated in Table 1, Classes I.b, I.c, and I.d are also equivalent to Class I, and hence Classes I.a–I.d are equivalent to each other. The "Derived Properties" listed in the top portion of Table 1 apply to all of the (equivalent) Classes I.a–I.d.

The equivalence of Classes I and I.b is easily established. From Table 1 we see that Class I.b is defined by choosing $\mathbb{D}_p = \tilde{D}_p$ and $\mathbb{W}_p = \tilde{W}_p$ in (3.4), resulting in flow rules of the form

Class I.b: $\tilde{D}_p = \mathcal{D}_p(\mathcal{S}),$ $\tilde{W}_p = \mathcal{W}_p(\mathcal{S}).$ (4.3)

On the other hand, for a flow rule in Class I we have $\dot{F}_p F_p^{-1} = \Phi_p(\mathcal{S})$, so it follows from the definitions (3.13) and (4.1) of $\tilde{D}_p$ and $\tilde{W}_p$ that these tensors depend only on C_e and the state variables $\mathcal{S}$; and since C_e is a state variable, we see that (4.3) holds. Therefore every flow rule in Class I belongs to Class I.b. Conversely, from the definitions (3.13) and (4.1) of $\tilde{D}_p$ and $\tilde{W}_p$ we see that

$$\tilde{D}_p + \tilde{W}_p = C_e \dot{F}_p F_p^{-1}, \tag{4.4}$$

and therefore

$$\dot{F}_p F_p^{-1} = C_e^{-1}\left(\tilde{D}_p + \tilde{W}_p\right). \tag{4.5}$$

Then for flow rules in Class I.b, the constitutive relations (4.3) and the kinematic identity (4.5) imply that $\dot{F}_p F_p^{-1}$ depends only on the state variables, which defines a flow rule in Class I. Thus Class I.b and Class I are equivalent. The proof that Classes I.c and I.d are equivalent to Class I is less straightforward and is given in Section 5.

4.2. Mutually exclusive classes of flow rules

From Table 1 we see that the classes of flow rules labeled II and III are characterized as follows:

Class II: $\qquad \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) = \mathcal{D}_p(\mathcal{S}), \qquad \Omega_p = \mathcal{W}_p(\mathcal{S}), \tag{4.6}$

Class III: $\qquad \tilde{D}_p = \mathcal{D}_p(\mathcal{S}), \qquad \Omega_p = \mathcal{W}_p(\mathcal{S}). \tag{4.7}$

In particular, for both classes the plastic spin tensor Ω_p depends only on the state variables.

We say that two classes of flow rules are *mutually exclusive* if they have no flow rules in common. In Section 6 we will prove that Classes I, II, and III are mutually exclusive. The details of the proof depend on the particular classes being compared, but the essential ideas are the same in each case. First, we show that there is a general kinematic relation of the form

$$\Omega_p = \hat{\Omega}_p\left(\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right), \mathrm{skw}\left(\dot{F}_p F_p^{-1}\right), V_p\right), \tag{4.8}$$

where the dependence on the left plastic stretch tensor $V_p = \sqrt{F_p F_p^T}$ is nontrivial. For flow rules in Class I, $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ depend only on the state variables $\mathcal{S}$, in which case (4.8) implies that the plastic spin tensor Ω_p necessarily depends not only on the state variables but also on V_p. Since, by assumption, V_p is not a state variable, such a flow rule does not belong to Classes II or III. Thus Classes I and II are mutually exclusive, as are Classes I and III. Next, we show that there is a general kinematic relation of the form

$$\tilde{D}_p = \tilde{D}_p^*\left(\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right), \Omega_p, C_e, V_p\right), \tag{4.9}$$

where the dependence on the V_p is nontrivial. For flow rules in Class II, $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and Ω_p depend only on the state variables $\mathcal{S}$, in which case (4.9)

implies that the plastic strain rate tensor $\tilde{D}_p$ depends on V_p as well as the state variables. By (4.7), such a flow rule does not belong to Class III. Therefore Classes II and III are mutually exclusive.

In Section 6 we also show that for flow rules in Class II, the preferred measures of plastic spin for Class I, namely $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ and $\tilde{W}_p$, depend on V_p as well as the state variables S. For flow rules in Class III, we show that the plastic spin tensors $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ and $\tilde{W}_p$, as well as the preferred measure of plastic strain rate for Class II, namely $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$, depend on V_p and the state variables S.[9]

From (4.6)$_1$ and the results in the previous paragraph, we see that for any flow rule in Classes II or III, $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ depend at most on the state variables S and the plastic stretch V_p. And since $\dot{F}_p F_p^{-1} = \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) + \mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$, it follows that flow rules in Classes II and III can be expressed in the form

$$\dot{F}_p F_p^{-1} = \hat{\Phi}_p(S, V_p) \tag{4.10}$$

for some function $\hat{\Phi}_p$. This implies that the flow rules in Classes II and III satisfy the third criterion in Section 3. Indeed, since $F_p = V_p R_p$, (4.10) is the special case of (3.2) with no dependence on the rotational part R_p of F_p. Of course, flow rules in Class I are special cases of (4.10) with no dependence on V_p.

Note that while every flow rule in Classes I–III can be expressed in the form (4.10), not every relation of the form (4.10) yields a flow rule in one of these classes. For flow rules in Class II, $\mathrm{sym}\hat{\Phi}_p$ depends only on the state variables S, and while $\mathrm{skw}\hat{\Phi}_p$ depends nontrivially on V_p, it does so in such a way that Ω_p is independent of V_p; the explicit form of this dependence is discussed in Section 6. For flow rules in Class III, both $\mathrm{sym}\hat{\Phi}_p$ and $\mathrm{skw}\hat{\Phi}_p$ depend nontrivially on V_p, but again this dependence is not arbitrary since $\tilde{D}_p$ and Ω_p must be independent of V_p.

4.3. Flow rules with null plastic spin

For flow rules in Classes II and III, the evolution equation for the plastic rotation R_p has the form $\dot{R}_p R_p^T = \Omega_p = \mathcal{W}_p(S)$. In particular, the spin function $\mathcal{W}_p$ may be chosen to be identically zero, in which case $\Omega_p \equiv 0$, which is equivalent to $\dot{R}_p \equiv 0$. Since we assume that F_p is initially equal to the identity tensor I, we also have $R_p = I$ initially. Therefore,

$$\Omega_p \equiv 0 \quad \text{iff} \quad R_p \equiv I \quad \text{iff} \quad F_p \equiv U_p \equiv V_p. \tag{4.11}$$

Classes II and III include as a special case flow rules for which the plastic spin tensor Ω_p is identically 0 for all motions, and for such flow rules there is no plastic rotation (as measured by R_p), and F_p is a pure stretch.

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[9] The proof of the results for Class III is more difficult and is given in Appendix B3.

Class I includes flow rules for which $\mathrm{skw}\!\left(\dot{F}_p F_p^{-1}\right)$ is identically 0 for all motions and also flow rules for which $\tilde{W}_p$ is identically 0 for all motions. These two sub-classes do not coincide in general. For a flow rule in Class I with $\mathrm{skw}\!\left(\dot{F}_p F_p^{-1}\right) \equiv 0$, the spin tensor $\tilde{W}_p$ will generally be nonzero, although it necessarily depends only on the state $\mathcal{S}$ since Classes I.a–I.d are equivalent. Similarly, for a flow rule in Class I with $\tilde{W}_p \equiv 0$, the spin tensor $\mathrm{skw}\!\left(\dot{F}_p F_p^{-1}\right)$ will generally be a nonzero function of the state $\mathcal{S}$. The conditions $\mathrm{skw}\!\left(\dot{F}_p F_p^{-1}\right) \equiv 0$ and $\tilde{W}_p \equiv 0$ are equivalent at some instant iff C_e and $\mathrm{sym}\!\left(\dot{F}_p F_p^{-1}\right)$ commute at that instant (see Section 5).

Given that Classes I, II, and III are mutually exclusive, it is easy to show that there are no flow rules in Class I for which Ω_p is identically 0 for all motions and that there are no flow rules in Classes II and III for which either $\mathrm{skw}\!\left(\dot{F}_p F_p^{-1}\right)$ or $\tilde{W}_p$ is identically 0 for all motions. Since $\mathrm{sym}\!\left(\dot{F}_p F_p^{-1}\right) = \mathcal{D}_p(\mathcal{S})$ for flow rules in Class I, a flow rule in that class for which $\Omega_p \equiv 0$ would satisfy the conditions $\mathrm{sym}\!\left(\dot{F}_p F_p^{-1}\right) = \mathcal{D}_p(\mathcal{S})$ and $\Omega_p \equiv 0$. But by (4.6), this is a flow rule in Class II with $\mathcal{W}_p(\mathcal{S}) \equiv 0$, contrary to the fact that Classes I and II are mutually exclusive. Thus, there are no flow rules in Class I for which Ω_p is identically 0 for all motions; equivalently, there are no flow rules in Class I for which $F_p \equiv U_p \equiv V_p$ for all motions. Similarly, since $\mathrm{sym}\!\left(\dot{F}_p F_p^{-1}\right) = \mathcal{D}_p(\mathcal{S})$ for flow rules in Class II and $\tilde{D}_p = \mathcal{D}_p(\mathcal{S})$ for flow rules in Class III, we see from (3.4) and Table 1 that a flow rule in Class II or III for which either $\mathrm{skw}\!\left(\dot{F}_p F_p^{-1}\right)$ or $\tilde{W}_p$ is identically 0 for all motions would also be a flow rule in one of the (equivalent) Classes I.a–I.d with $\mathcal{W}_p(\mathcal{S}) \equiv 0$, contrary to the fact that Classes I and II and Classes I and III are mutually exclusive.

4.4. The plastic stress power

In Section 3 we observed that for flow rules in Class I, the plastic stress power depends only on the state variables $\mathcal{S}$. Therefore, this property holds for each of the (equivalent) Classes I.a–I.d. We also observed that this property holds for any flow rule for which $\tilde{D}_p$ depends only on the state variables. In particular, it holds for all flow rules in Class III. Finally, we noted that if the material is elastically isotropic, then the plastic stress power depends only on the state variables for any flow rule for which $\mathrm{sym}\!\left(\dot{F}_p F_p^{-1}\right)$ depends only on the state variables. In particular, for elastically isotropic materials the plastic stress power depends only on the state variables for all flow rules in Class II.

For flow rules in Class II and materials that are not elastically isotropic, the plastic stress power will generally depend on the plastic stretch V_p as well as the state variables $\mathcal{S}$. To see this, note that if the tensors $C_e \tilde{T}$ and $\dot{F}_p F_p^{-1}$ are decomposed into

their symmetric and skew parts, then the expression (3.1) for the plastic stress power becomes

$$\rho_R \mathcal{P}_p^s = \mathrm{sym}\left(C_e \tilde{T}\right) : \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) + \mathrm{skw}\left(C_e \tilde{T}\right) : \mathrm{skw}\left(\dot{F}_p F_p^{-1}\right). \tag{4.12}$$

For flow rules in Class II, $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ depends only on the state variables, and since C_e and $\tilde{T}$ are state variables, the first group of terms on the right depends only on the state variables. However, as noted above, for flow rules in Class II the plastic spin $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ depends on V_p as well as the state variables. If $\mathrm{skw}\left(C_e \tilde{T}\right) \neq 0$ (equivalently, if C_e and $\tilde{T}$ do not commute), as is generally the case, then $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ will generally make a nonzero contribution to the the plastic stress power, and thus the plastic stress power will generally depend on the plastic stretch V_p as well as the state variables.

4.5. Generalized plastic strain rate tensors

Generalized elastic strain tensors E_e^f, their conjugate stress tensors $\tilde{T}_f$, and the plastic strain rate tensors $\tilde{D}_p^f$ conjugate to $\tilde{T}_f$ were discussed by Scheidler and Wright (2001, Section 4.10). Following Hill (1978),[10] a smooth real-valued function f defined on the positive reals and satisfying the conditions $f(1)=0$, $f'(1)=1$, and $f' > 0$ may be regarded as a scalar strain measure. The elastic strain tensor E_e^f corresponding to the strain measure f is the symmetric tensor that is coaxial with U_e but with corresponding eigenvalues $f(\lambda_i^e)$, where λ_i^e are the eigenvalues of U_e, i.e., the principal elastic stretches. The stress tensor $\tilde{T}_f$ conjugate to E_e^f satisfies the requirement that $\tilde{T}_f : \dot{E}_e^f$ is the elastic stress power per unit volume in the intermediate configuration; equivalently, $\tilde{T}_f : \dot{E}_e^f = \tilde{T} : \dot{E}_e$. The plastic strain rate tensor $\tilde{D}_p^f$ conjugate to $\tilde{T}_f$ satisfies the requirement that $\tilde{T}_f : \tilde{D}_p^f$ is the plastic stress power per unit volume in the intermediate configuration; equivalently, $\tilde{T}_f : \tilde{D}_p^f = \tilde{T} : \tilde{D}_p$.[11] In particular, when $f(\lambda) = \frac{1}{2}\left(\lambda^2 - 1\right)$, we have $E_e^f = E_e$, $\tilde{T}_f = \tilde{T}$, and $\tilde{D}_p^f = \tilde{D}_p$.

Instead of assuming that $\tilde{D}_p = \mathcal{D}_p(\mathcal{S})$ as in Classes III, I.b and I.d, we could consider flow rules for which

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[10] See also Ogden (1984) and Scheidler (1991). All of these authors considered total strain tensors rather than their elastic part as considered here, but the ideas are essentially the same in either case.

[11] The stress tensor $\tilde{T}_f$ conjugate to E_e^f is given by $\tilde{T}_f = \mathbb{F}_{U_e}\left[\tilde{T}\right] \equiv Df(U_e)^{-1}\left[\mathrm{sym}\left(U_e \tilde{T}\right)\right]$, where the fourth order tensor $Df(U_e)$ is the derivative of the function $f(U_e)$ whose value is E_e^f. Then $\tilde{D}_p^f = \mathbb{F}_{U_e}^{-1}\left[\tilde{D}_p\right]$, and it can be shown that $\mathbb{F}_{U_e}^{-1} = 2Df(U_e)\mathbb{L}_{U_e} = 2\mathbb{L}_{U_e}Df(U_e)$, where the fourth-order tensor $\mathbb{L}_{U_e}$ is defined in Appendix A. Except for the special cases corresponding to $f(\lambda) = \frac{1}{m}(\lambda^m - 1)$ with m an integer, explicit component-free formulas for the above results are rather complicated. In general, it is simpler to express these results in component form relative to a principal basis for U_e. Component formulas for $Df(U_e)$ can be found in Ogden (1984, p. 162), and Scheidler (1991). A component formula for $\mathbb{L}_{U_e}$ follows from Eq. (A.2) in Appendix A.

$$\tilde{D}_p^f = \mathcal{D}_p^f(\mathcal{S}_f), \qquad \text{where} \qquad \mathcal{S}_f = \left(E_e^f, \theta, q_n, A_k\right) \text{ or } \left(\tilde{T}_f, \theta, q_n, A_k\right). \qquad (4.13)$$

However, as observed in Scheidler and Wright (2001, Section 5.1), a relation of the form (4.13) is equivalent to a relation of the form $\tilde{D}_p = \mathcal{D}_p(\mathcal{S})$, so (4.13) does not lead to anything new as far as the classifications in this section are concerned. In particular, Class III and the equivalent Classes I.b and I.d remain unchanged if the relation $\tilde{D}_p = \mathcal{D}_p(\mathcal{S})$ is replaced by (4.13).

4.6. Isotropic materials with only scalar internal variables

For an isotropic material with only scalar internal variables, the flow functions $\mathcal{D}_p$ and $\mathcal{W}_p$ in the flow rule (3.4) are isotropic functions of the state $\mathcal{S}$, which in this case contains only one tensor state variable, e.g.,

$$\mathcal{S} = (E_e, \theta, q_n) \quad \text{or} \quad \left(\tilde{T}, \theta, q_n\right). \qquad (4.14)$$

Since the values of an isotropic tensor-valued function of a single symmetric tensor (and an arbitrary number of scalars) are necessarily symmetric, and since $\mathcal{W}_p$ is skew-valued by assumption, it follows that the function $\mathcal{W}_p$ is identically zero in this case.[12] Thus, for an isotropic material with only scalar internal variables, $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ and $\tilde{W}_p$ are identically zero for flow rules in Class I, whereas Ω_p is identically zero for flow rules in Classes II and III:

Class I: $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) = \mathcal{D}_p(\mathcal{S}), \qquad \mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) \equiv 0, \qquad (4.15)$

Class II: $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) = \mathcal{D}_p(\mathcal{S}), \qquad \Omega_p \equiv 0, \qquad (4.16)$

Class III: $\tilde{D}_p = \tilde{D}_p(\mathcal{S}), \qquad \Omega_p \equiv 0. \qquad (4.17)$

Even for this special case, Classes I, II, and III are mutually exclusive. In particular, $R_p \equiv I$ and $F_p \equiv U_p \equiv V_p$ for flow rules in Classes II and III [see (4.11)]. However, for flow rules in Class I, R_p is generally not fixed and depends (in part) on the history of V_p, since Ω_p is generally nonzero and depends on V_p.

In spite of these differences in the evolution of the intermediate configuration, it can be shown that for isotropic materials with only scalar internal variables, the flow rules (4.15) and (4.16) characterizing Classes I and II, respectively, produce the same Cauchy stress for a given flow function $\mathcal{D}_p$, other things being equal.[13] This is generally not the case if the material is anisotropic or if tensor internal variables are included. This issue and others related to material symmetry will be addressed in a follow-up paper.

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[12] This argument is well-known; for example, see Kratochvil (1973), who applied it to flow rules in our Class I.

[13] See the Appendix in Boyce et al. (1989), with their back stress B set to zero.

5. Equivalence of Classes I.a, I.c, and I.d

As indicated in Table 1, the Classes I.a–I.d are equivalent to Class I [as defined by (3.3)], and hence they are equivalent to each other. We have already established the equivalence of Classes I, I.a, and I.b (see Sections 3.3 and 4.1). In this section we show that Classes I.a, I.c, and I.d are equivalent. At the end of the section we discuss the contribution of the plastic spin to the plastic stress power for flow rules in Class I.

5.1. Proof that Class I.a is contained in Classes I.c and I.d

The plastic strain rate tensor $\tilde{D}_p$ and plastic spin tensor $\tilde{W}_p$ are given by

$$\tilde{D}_p = \text{sym}\left(C_e \dot{F}_p F_p^{-1}\right) = \frac{1}{2}\left[C_e\left(\dot{F}_p F_p^{-1}\right) + \left(\dot{F}_p F_p^{-1}\right)^T C_e\right] \tag{5.1}$$

and

$$\tilde{W}_p = \text{skw}\left(C_e \dot{F}_p F_p^{-1}\right) = \frac{1}{2}\left[C_e\left(\dot{F}_p F_p^{-1}\right) - \left(\dot{F}_p F_p^{-1}\right)^T C_e\right]. \tag{5.2}$$

If we expand $\dot{F}_p F_p^{-1}$ into symmetric and skew parts in $(5.1)_2$ and $(5.2)_2$, we obtain the relations

$$2\tilde{D}_p = C_e\text{sym}\left(\dot{F}_p F_p^{-1}\right) + \text{sym}\left(\dot{F}_p F_p^{-1}\right)C_e + C_e\text{skw}\left(\dot{F}_p F_p^{-1}\right)$$
$$- \text{skw}\left(\dot{F}_p F_p^{-1}\right)C_e \tag{5.3}$$

and

$$2\tilde{W}_p = C_e\text{sym}\left(\dot{F}_p F_p^{-1}\right) - \text{sym}\left(\dot{F}_p F_p^{-1}\right)C_e + C_e\text{skw}\left(\dot{F}_p F_p^{-1}\right)$$
$$+ \text{skw}\left(\dot{F}_p F_p^{-1}\right)C_e. \tag{5.4}$$

For a flow rule in Class I.a, $\text{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\text{skw}\left(\dot{F}_p F_p^{-1}\right)$ depend only on the state variables, and thus, by (5.3)–(5.4), so do $\tilde{D}_p$ and $\tilde{W}_p$, since C_e is a state variable. It follows that a flow rule in Class I.a is also in Classes I.c and I.d, as defined in Table 1. The same conclusion also follows from (5.1)–(5.2) and (3.3).

5.2. Proof that Classes I.c and I.d are contained in Class I.a

To prove the reverse implications, we solve (5.3) for $\text{sym}\left(\dot{F}_p F_p^{-1}\right)$ and (5.4) for $\text{skw}\left(\dot{F}_p F_p^{-1}\right)$. The results may be expressed as

$$\text{sym}\left(\dot{F}_p F_p^{-1}\right) = 2\mathbb{L}_{C_e}\left[\tilde{D}_p\right] - \mathbb{M}_{C_e}\left[\text{skw}\left(\dot{F}_p F_p^{-1}\right)\right], \tag{5.5}$$

$$\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) = 2\mathbb{L}_{C_e}\left[\tilde{W}_p\right] - \mathbb{M}_{C_e}\left[\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right]$$

(5.6)

The derivation of these solutions is given in Appendix B1. The symbols $\mathbb{L}_{C_e}$ and $\mathbb{M}_{C_e}$ denote certain fourth-order tensors that depend (nonlinearly) on C_e. More generally, for any second-order, symmetric positive-definite tensor A, the symbols $\mathbb{L}_A$ and $\mathbb{M}_A$ denote certain fourth-order tensors that depend (nonlinearly) on A.[14] The definitions of $\mathbb{L}_A$ and $\mathbb{M}_A$ are given in Appendix A, along with a discussion of some of their properties. Although the only case of interest in this section is $A = C_e$, the case $A = V_p$ arises in the next section. For a given A, the fourth-order tensors $\mathbb{L}_A$ and $\mathbb{M}_A$ are regarded as linear transformations on the space of second-order tensors. The image of a second-order tensor H under the mapping $\mathbb{M}_A$, for example, is a second-order tensor denoted by $\mathbb{M}_A[H]$. Explicit formulas for $\mathbb{M}_A[H]$ and $\mathbb{L}_A[H]$ are also given in Appendix A. These formulas yield explicit expressions for the kinematic relations (5.5) and (5.6).

Here we are concerned primarily with the qualitative properties of $\mathbb{L}_A$ and $\mathbb{M}_A$. We note that $\mathbb{L}_A$ maps symmetric tensors to symmetric tensors and skew tensors to skew tensors, whereas $\mathbb{M}_A$ maps skew tensors to symmetric tensors and symmetric tensors to skew tensors. Thus, each of the two groups of terms on the right side of (5.5) is symmetric, whereas each of the two groups of terms on the right side of (5.6) is skew.

For a flow rule in Class I.c, $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\tilde{W}_p$ depend only on the state S (by assumption), and thus, by (5.6), so does $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$. That is, both $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ depend only on S, so the flow rule is in Class I.a. Similarly, for a flow rule in Class I.d, $\tilde{D}_p$ and $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ depend only on S (by assumption), and thus by (5.5) so does $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$, so again the flow rule is in Class I.a. Therefore, the classes of flow rules I.a, I.c, and I.d are equivalent.

5.3. Further remarks on the preferred plastic spins for flow rules in Class I

From (5.4) it follows that the condition $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) = 0$ does not imply $\tilde{W}_p = 0$, in general. Similarly, from (5.6) it follows that the condition $\tilde{W}_p = 0$ does not imply $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) = 0$, in general. On the other hand, from (5.4) we see that

$$\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) = 0 \iff \tilde{W}_p = 0, \quad \text{if } \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) \text{ and } C_e \text{ commute.}$$

(5.7)

This result may also be obtained from (5.6), since $\mathbb{L}_{C_e}\left[\tilde{W}_p\right] = 0$ iff $\tilde{W}_p = 0$, and $\mathbb{M}_{C_e}\left[\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right] = 0$ iff C_e and $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ commute (see Appendix A).

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[14] In this context the tensor A is unrelated to the tensor internal variables A_k.

Recall that Class I can be characterized by flow rules of the form $\mathbb{D}_p = \mathcal{D}_p(\mathcal{S})$ and $\mathbb{W}_p = \mathcal{W}_p(\mathcal{S})$ with either $\mathbb{D}_p = \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\mathbb{W}_p = \mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ (Class I.a), or $\mathbb{D}_p = \tilde{D}_p$ and $\mathbb{W}_p = W_p$ (Class I.b). Then the relations (4.12) and (3.12) for the plastic stress power might suggest that for flow rules in Class I, $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ generally contributes to the plastic stress power whereas $\tilde{W}_p$ does not. Actually, this conclusion is not correct without some qualification. The correct interpretation is that $\tilde{D}_p$ and $\tilde{T}$ completely determine the plastic stress power, whereas $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\tilde{T}$ (or even $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$, $\tilde{T}$, and C_e) generally do not, so that if $\mathbb{D}_p = \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ then the the additional contribution to the plastic stress power must be provided by the preferred spin $\mathbb{W}_p$. For example, from Table 1 we see that Class I is also characterized by $\mathbb{D}_p = \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\mathbb{W}_p = W_p$ (Class I.c). Then (4.12) and the formula (5.6) for $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ in terms of $\tilde{W}_p$ and $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ imply that the plastic stress power can be expressed in terms of $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$, $\tilde{W}_p$, C_e, and $\tilde{T}$; in particular, $\tilde{W}_p$ generally makes a nonzero contribution to this expression. On the other hand, from Table 1 we see that Class I is also characterized by $\mathbb{D}_p = \tilde{D}_p$ and $\mathbb{W}_p = \mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ (Class I.d); and by (3.12), $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ makes no contribution to the expression $\tilde{T} : \tilde{D}_p$ for the plastic stress power.

6. Proof that Classes I, II, and III are mutually exclusive

In this section we prove that Classes I, II, and III are mutually exclusive. The proof is based on a fundamental kinematic identity that relates the plastic spin tensors $\mathbf{\Omega}_p$ and $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$, the plastic strain rate tensor $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$, and the plastic stretch tensor V_p:

$$\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) - \mathbf{\Omega}_p = \mathbb{M}_{V_p}\left[\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right]. \tag{6.1}$$

In particular, (6.1) implies that $\mathbf{\Omega}_p$ is a function of $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$, $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$, and V_p, as alluded to in (4.8). Recall that $\mathbb{M}_{V_p}$ is a fourth-order tensor depending (nonlinearly) on V_p, as defined in Appendix A, and that $\mathbb{M}_{V_p}$ maps symmetric tensors to skew tensors, so that the right-hand side of (6.1) is indeed a skew tensor as required.

6.1. Derivation of the kinematic relation (6.1)

There are several ways to arrive at the identity (6.1). The method we use here is based on the following two properties (see Appendix A). First, $\mathbb{M}_A[H]$ is an isotropic function of A and H:

$$Q\mathbb{M}_A[H]Q^T = \mathbb{M}_{QAQ^T}[QHQ^T] \tag{6.2}$$

for any orthogonal tensor Q. Second, for any symmetric tensor B,

$$\mathrm{skw}(BA^{-1}) = \mathbb{M}_A[\mathrm{sym}(BA^{-1})]. \tag{6.3}$$

In particular, the skew part of BA^{-1} is determined by A and the symmetric part of BA^{-1}. Recall that A is assumed to be symmetric and positive-definite.

From the polar decomposition $F_p = R_p U_p$ and the definition $\Omega_p = \dot{R}_p R_p^T$, we obtain the kinematic relation

$$\dot{F}_p F_p^{-1} = R_p (\dot{U}_p U_p^{-1}) R_p^T + \Omega_p. \tag{6.4}$$

On taking the symmetric and skew parts of this relation, we obtain

$$\mathrm{sym}(\dot{F}_p F_p^{-1}) = R_p \mathrm{sym}(\dot{U}_p U_p^{-1}) R_p^T, \tag{6.5}$$

$$\mathrm{skw}(\dot{F}_p F_p^{-1}) = R_p \mathrm{skw}(\dot{U}_p U_p^{-1}) R_p^T + \Omega_p. \tag{6.6}$$

By setting $B = \dot{U}_p$ and $A = U_p$ in the algebraic identity (6.3), we obtain a compatibility condition for $\mathrm{skw}(\dot{U}_p U_p^{-1})$ due to Nemat-Nasser (1990, 1992),[15]

$$\mathrm{skw}(\dot{U}_p U_p^{-1}) = \mathbb{M}_{U_o}[\mathrm{sym}(\dot{U}_p U_p^{-1})]. \tag{6.7}$$

On substituting this relation into (6.6) and then using (6.2) with $A = U_p$ and $Q = R_p$, we obtain

$$\begin{aligned}
\mathrm{skw}(\dot{F}_p F_p^{-1}) - \Omega_p &= R_p \mathrm{skw}(\dot{U}_p U_p^{-1}) R_p^T \\
&= R_p \mathbb{M}_{U_p}[\mathrm{sym}(\dot{U}_p U_p^{-1})] R_p^T \\
&= \mathbb{M}_{R_p U_p R_p^T}[R_p \mathrm{sym}(\dot{U}_p U_p^{-1}) R_p^T].
\end{aligned}$$

On using $R_p U_p R_p^T = V_p$ and the relation (6.5) in the bottom expression, we obtain (6.1). An alternate derivation of (6.1) is given in Appendix B2.

Explicit formulas for $\mathbb{M}_A[H]$ are given in Appendix A. On setting $A = V_p$ and $H = \mathrm{sym}(\dot{F}_p F_p^{-1})$ in the formulas (A.5) and (A.17) for $\mathbb{M}_A[H]$ and then substituting the results into (6.1) above, we can obtain explicit formulas for the difference $\mathrm{skw}(\dot{F}_p F_p^{-1}) - \Omega_p$ in terms of $\mathrm{sym}(\dot{F}_p F_p^{-1})$ and V_p. For the discussion in this section, only the qualitative properties of $\mathbb{M}_{V_p}$ are needed.

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[15] For the special case where $F_p \equiv U_p \equiv V_p$, (6.7) was also obtained by Obata et al. (1990).

6.2. Proof that Classes I–III are mutually exclusive

For flow rules in Class I or Class II, we have $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) = \mathcal{D}_p(\mathcal{S})$, so that by (6.1),

Classes I & II: $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) - \Omega_p = \mathbb{M}_{V_p}[\mathcal{D}_p(\mathcal{S})].$ (6.8)

Hence, for flow rules in Class I or Class II, the difference $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) - \Omega_p$ depends not only on the state variables $\mathcal{S}$ but also on the left plastic stretch tensor V_p. This dependence on V_p is nontrivial. Indeed, as discussed in Appendix A, for a given tensor H the tensor $\mathbb{M}_{V_p}[H]$ is independent of the value of V_p iff H is spherical, in which case $\mathbb{M}_{V_p}[H] = 0$. Since $\mathcal{D}_p(\mathcal{S}) = \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ is deviatoric, it is spherical iff it is zero.

Since the difference $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) - \Omega_p$ depends on V_p as well as the state variables, it follows that Ω_p must depend on V_p as well as the state variables for a flow rule in Class I, since $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ depends only on the state variables. Such a flow rule does not belong to Classes II or III since, by assumption, Ω_p depends only on the state variables for flow rules in these classes. Thus Classes I and II are mutually exclusive, as are Classes I and III.

When the kinematic relation (6.1) is solved for $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ and the result is substituted into the relations (5.3)–(5.4), we obtain kinematic relations of the form

$$\tilde{D}_p = \tilde{D}_p^*\left(\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right), \Omega_p, C_e, V_p\right),$$

$$\tilde{W}_p = \tilde{W}_p^*\left(\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right), \Omega_p, C_e, V_p\right),$$ (6.9)

where the dependence on V_p is nontrivial. Now consider a flow rule in Class II. By assumption, $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and Ω_p depend only on the state variables. Then by (6.9) we see that $\tilde{D}_p$ and $\tilde{W}_p$ also depend on V_p as well as the state variables. In particular, since $\tilde{D}_p$ depends on V_p, the flow rule cannot belong to Class III. Thus Classes II and III are mutually exclusive. Also note that by (6.8), $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ depends on V_p as well as the state variables for flow rules in Class II.

Since $\dot{F}_p F_p^{-1} = \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) + \mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$, the kinematic relation (6.1) implies that

$$\dot{F}_p F_p^{-1} = \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) + \mathbb{M}_{V_p}\left[\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right] + \Omega_p.$$ (6.10)

This kinematic relation expresses the plastic velocity gradient in terms of V_p and the preferred measures for plastic strain rate and plastic spin for flow rules in Class II. Since these measures are functions of the state variables $\mathcal{S}$ for flow rules in this class, it follows that $\dot{F}_p F_p^{-1} = \hat{\Phi}_p(\mathcal{S}, V_p)$ for flow rules in Class II as claimed in Section 4.2, and hence that the third criterion in Section 3 is satisfied. The analogous result for flow rules in Class III is more difficult to establish; the proof is given in Appendix B3.

6.3. Further remarks on the plastic spin tensors $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ and Ω_p

As discussed in Section 4.3, Classes II and III contain a subclass of flow rules for which $\Omega_p \equiv 0$. The above results imply that there are no such flow rules in Class I. Indeed, if a flow rule in Class I satisfied $\Omega_p \equiv 0$, then by (6.8) we would have $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) = \mathbb{M}_{V_p}[\mathcal{D}_p(\mathcal{S})]$ for all motions, contrary to the fact that $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ depends only on the state variables for flow rules in Class I.

Since $\mathbb{M}_A[H] = 0$ iff A and H commute (see Appendix A), the kinematic identity (6.1) implies that

$$\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) = \Omega_p \iff V_p \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) = \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) V_p. \tag{6.11}$$

In other words, the plastic spin tensors $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ and Ω_p coincide at an instant iff $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and V_p commute at that instant. If the relation $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) = \Omega_p$ were required to hold for all possible motions of a material, then $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ would have to commute with V_p at every instant, which implies a dependence of $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ on V_p. For flow rules in Classes I or II, such dependence is not permitted, by assumption. Thus there are no flow rules in Classes I or II for which $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) = \Omega_p$ for all motions.

We close this section by establishing some bounds on the plastic spin tensors $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ and Ω_p that will be used in the next section. The norm of a tensor H is the scalar $\|H\| \geqslant 0$ defined by

$$\|H\|^2 = \mathrm{tr}\left(H^T H\right) = \sum_{i,j=1}^{3} H_{ij}^2, \tag{6.12}$$

where the components H_{ij} of H are taken relative to any orthonormal basis. Since $\left\|\mathbb{M}_{V_p}\left[\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right]\right\| \leqslant \left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\|$, [see (A.7)], the identity (6.1) yields the inequalities

$$\left\|\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) - \Omega_p\right\| \leqslant \left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\|, \tag{6.13}$$

$$\left\|\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)\right\| \leqslant \left\|\Omega_p\right\| + \left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\|, \tag{6.14}$$

$$\left\|\Omega_p\right\| \leqslant \left\|\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)\right\| + \left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\|. \tag{6.15}$$

Observe that unlike (6.1), these inequalities are independent of the plastic stretch V_p.

7. Approximations for small elastic shear strains

The results discussed up to this point and summarized in Table 1 are valid for arbitrary finite elastic deformations. Large elastic rotations cannot be ruled out since a superposed rigid motion is regarded as part of the elastic deformation only (Scheidler and Wright, 2001, Section 3.2). Also, high pressures, such as those generated by ballistic impact, result in large elastic volumetric strains, and it is known that polymers may sustain large elastic shear strains.[16] On the other hand, the elastic shear strains that can be sustained by metals are relatively small. In this section we discuss the validity of certain approximations for the plastic strain rate and plastic spin under the assumption that the elastic shear strains are small.

7.1. Measures of elastic shear strain

The elastic deformation gradient F_e can be decomposed into a dilatational part, $(\det F_e)^{1/3}$, and an isochoric part $\bar{F}_e$,

$$F_e = (\det F_e)^{1/3} \bar{F}_e = \left(\frac{\rho_R}{\rho}\right)^{1/3} \bar{F}_e, \qquad \det \bar{F}_e = 1. \tag{7.1}$$

Similarly, the elastic stretch tensors can be decomposed into dilatational and isochoric parts,

$$U_e = \left(\frac{\rho_R}{\rho}\right)^{1/3} \bar{U}_e, \qquad V_e = \left(\frac{\rho_R}{\rho}\right)^{1/3} \bar{V}_e, \qquad \det \bar{U}_e = \det \bar{V}_e = 1. \tag{7.2}$$

Then $\bar{F}_e$ has the polar decomposition

$$\bar{F}_e = R_e \bar{U}_e = \bar{V}_e R_e, \tag{7.3}$$

and

$$C_e = \left(\frac{\rho_R}{\rho}\right)^{2/3} \bar{C}_e, \qquad \bar{C}_e = \bar{F}_e^T \bar{F}_e = \bar{U}_e^2, \qquad \det \bar{C}_e = 1. \tag{7.4}$$

The symmetric positive-definite tensors $\bar{U}_e$ and $\bar{V}_e$ are independent of the dilatational and rotational parts of F_e and thus are measures of elastic distortion only. It follows that $\bar{U}_e - I$ and $\bar{V}_e - I$ can be regarded as tensor measures of elastic shear strain. The scalar $\|\bar{U}_e - I\| = \|\bar{V}_e - I\|$ measures the magnitude of the elastic shear strain, independent of the elastic volumetric strain and the elastic rotation. Likewise, the tensor

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[16] In this regard, see the recent paper by Srinivasa (2001) in which the effect of large elastic shear strains on plastic flow in simple shear is studied.

$$E_e = \frac{1}{2}(C_e - I) = (U_e - I) + \frac{1}{2}(U_e - I)^2 \qquad (7.5)$$

and the scalar $\|E_e\|$ may be regarded as measures of elastic shear strain.[17]

7.2. Approximations for small elastic shear strains

Plastic flow in metals limits the magnitude of the elastic shear strain so that $\|U_e - I\|$ is small compared to unity. Then $\|U_e - I\|^2$ is small compared to $\|U_e - I\|$, and so on. By an approximation for small elastic shear strains we mean any approximation obtained by neglecting these relatively small terms. For example, since we may neglect the term $U_e - I$ relative to I for small elastic shear strains, and since $U_e = I + (U_e - I)$, on using $(7.2)_1$ we obtain the approximations

$$U_e H \approx H U_e \approx \left(\frac{\rho_R}{\rho}\right)^{1/3} H \qquad (7.6)$$

for any tensor H.

From (7.5) we see that $\|E_e\|$ is small compared to unity iff $\|U_e - I\|$ is small compared to unity. Since we may neglect the term E_e relative to I for small elastic shear strains, and since

$$C_e = I + 2E_e, \qquad (7.7)$$

from $(7.4)_1$ we obtain the approximations

$$C_e H \approx H C_e \approx \left(\frac{\rho_R}{\rho}\right)^{2/3} H \qquad (7.8)$$

for any tensor H. On multiplying (7.7) by C_e^{-1} on the left or the right, we obtain the approximations

$$C_e^{-1} H \approx H C_e^{-1} \approx \left(\frac{\rho_R}{\rho}\right)^{-2/3} H \qquad (7.9)$$

for any tensor H.

By setting $H = \dot{F}_p F_p^{-1}$ in (7.8), we obtain the approximation

$$C_e \dot{F}_p F_p^{-1} \approx \left(\frac{\rho_R}{\rho}\right)^{2/3} \dot{F}_p F_p^{-1} \qquad (7.10)$$

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[17] If $\|E_e\|$ is sufficiently small and if, for example, the material is elastically isotropic, then $\mathrm{dev}\,T \approx 2\mu E_e$, where $\mathrm{dev}\,T$ denotes the deviatoric part of T and μ is the elastic shear modulus. However, even though the elastic shear strains may be small, μ cannot be assumed constant as in the linear theory of elasticity. The elastic shear modulus generally depends on both the temperature and the pressure. Constitutive relations for this dependence are discussed in the recent paper by Hanim and Ahzi (2001).

for small elastic shear strains. Now by (4.4) we have $C_e\dot{F}_pF_p^{-1} = \tilde{D}_p + \tilde{W}_p$. On substituting this result into the left side of (7.10), we obtain the approximation

$$\dot{F}_pF_p^{-1} \approx \left(\frac{\rho_R}{\rho}\right)^{-2/3}\left(\tilde{D}_p + \tilde{W}_p\right) \tag{7.11}$$

for small elastic shear strains. This also follows from the relation (4.5) for $\dot{F}_pF_p^{-1}$ by setting $H = \tilde{D}_p + \tilde{W}_p$ in the approximate formula (7.9).

Next, we give two typical (but erroneous) arguments that apparently lead to the approximations

$$\tilde{D}_p \approx \left(\frac{\rho_R}{\rho}\right)^{2/3}\mathrm{sym}\left(\dot{F}_pF_p^{-1}\right), \qquad \tilde{W}_p \approx \left(\frac{\rho_R}{\rho}\right)^{2/3}\mathrm{skw}\left(\dot{F}_pF_p^{-1}\right) \tag{7.12}$$

for small elastic shear strains. First, since $\tilde{D}_p = \mathrm{sym}\left(C_e\dot{F}_pF_p^{-1}\right)$ and $\tilde{W}_p = \mathrm{skw}\left(C_e\dot{F}_pF_p^{-1}\right)$, it seems reasonable to conclude that the approximations (7.12) follow by taking the symmetric and skew parts of the approximate relation (7.10) for $C_e\dot{F}_pF_p^{-1}$. Alternatively, the approximations (7.12) would also seem to follow by taking symmetric and skew parts of the approximate relation (7.11) for $\dot{F}_pF_p^{-1}$. However, as shown in the next subsection, *the assumption of small elastic shear strains is not, by itself, sufficient for the validity of the approximations* (7.12). In the next subsection we will derive correct approximations for $\tilde{D}_p$ and $\tilde{W}_p$ and from these determine additional restrictions that suffice for the validity of the simpler approximations (7.12).

The importance of the approximation (7.12)₁ stems from the following observation. Suppose the material is such that the approximation (7.12)₁ for $\tilde{D}_p$ is valid whenever the elastic shear strains are small. In this case we see that if $\mathrm{sym}\left(\dot{F}_pF_p^{-1}\right)$ depends only on the state variables S then, approximately, so does $\tilde{D}_p$, since $(\rho_R/\rho)^{2/3} = (\det C_e)^{1/3}$ is a state variable. Likewise, if $\tilde{D}_p$ depends only on the state variables then, approximately, so does $\mathrm{sym}\left(\dot{F}_pF_p^{-1}\right)$. *It follows that for small elastic shear strains, Classes II and III may be regarded as approximately equivalent provided the approximation* (7.12)₁ *is valid.* In view of this result and the fact that Classes II and III are mutually exclusive for finite elastic deformations, it is of some interest to establish the conditions under which the approximation (7.12)₁ for $\tilde{D}_p$ does hold. As noted above, the assumption of small elastic shear strains is not sufficient by itself.

Note that the approximate equivalence of Classes II and III does not extend to Class I. For small elastic shear strains, Class I still has no flow rules in common with Classes II and III, even when the approximation (7.12)₁ for $\tilde{D}_p$ is valid. This is due to the fact that the kinematic identity (6.1) does not involve the elastic strain and thus does not simplify for small elastic shear strains, so that the difference between $\mathrm{skw}\left(\dot{F}_pF_p^{-1}\right)$ and Ω_p still depends on V_p.

7.3. Correct approximations for $\tilde{D}_p$ and $\tilde{W}_p$

Before deriving the correct approximations for $\tilde{D}_p$ and $\tilde{W}_p$, we give a simple geometric explanation for the failure of the approximations (7.12). Consider a vector space Y and two vectors u and v that are approximately equal. Let u^* and v^* denote the projections of u and v onto a subspace Y^* of Y that is nearly orthogonal to u. For the case illustrated in Fig. 1, the length of v^* is twice that of u^*. Thus, even if the relative error in approximating u by v is small, the relative error in approximating the projection of u by the projection of v may be large. Now let Y be the vector space of second-order tensors, and let Y^* be the subspace Sym of symmetric tensors. If we take $u = (\rho_R/\rho)^{2/3}\dot{F}_p F_p^{-1}$ and $v = C_e\dot{F}_p F_p^{-1}$, then $u^* = (\rho_R/\rho)^{2/3}\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $v^* = \mathrm{sym}\left(C_e\dot{F}_p F_p^{-1}\right) = \tilde{D}_p$ are the projections of u and v onto the subspace Y^*. By (7.10), u and v are approximately equal for small elastic shear strains. But from the above discussion we see that the relative error in approximating u^* by v^* may be large. In other words, the approximation (7.12)$_1$ for $\tilde{D}_p$ is not valid in general. This approximation fails when the projection of $\dot{F}_p F_p^{-1}$ onto the orthogonal complement of Sym, namely the skew part of $\dot{F}_p F_p^{-1}$, is sufficiently large relative to the symmetric part. Similarly, by taking Y^* to be the subspace of skew tensors, we see that the approximation (7.12)$_2$ for $\tilde{W}_p$ fails when the symmetric part of $\dot{F}_p F_p^{-1}$ is sufficiently large relative to the skew part.

Assuming only small elastic shear strains, the correct approximations for $\tilde{D}_p$ and $\tilde{W}_p$ are

$$\tilde{D}_p \approx \left(\frac{\rho_R}{\rho}\right)^{2/3}\left[\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) + E_e\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) - \mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)E_e\right] \qquad (7.13)$$

and

$$\tilde{W}_p \approx \left(\frac{\rho_R}{\rho}\right)^{2/3}\left[\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) + E_e\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) - \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)E_e\right]. \qquad (7.14)$$

To obtain (7.13), first note that by (7.4)$_1$ and (7.7), we have $C_e = (\rho_R/\rho)^{2/3}(I + 2E_e)$. When this is substituted into the expression (5.3) for $\tilde{D}_p$, we obtain the expression on the right-hand side of (7.13) plus an additional expression (inside the brackets) which is relatively small. Indeed, the omitted expression is $E_e\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) + \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)E_e$. Since

$$\|E_e H \pm H E_e\| \leqslant 2\|E_e\| \cdot \|H\| \qquad (7.15)$$

for any tensor H, the omitted expression is bounded by $2\|E_e\| \cdot \left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\|$. For small elastic shear strains, this is small relative to the term $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ in (7.13), and the approximation (7.13) follows. The approximation (7.14) is derived similarly, starting from the expression (5.4) for $\tilde{W}_p$.

Now it is certainly true that (7.13) and (7.14) yield the approximations (7.12) in the limit as E_e approaches **0**. In other words, for any given values of $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$, the approximations (7.12) hold for sufficiently small values of E_e. However, we are not free to make the elastic shear strain tensor E_e as small as desired. For a given material model, the values taken on by E_e are determined by the stress–temperature or strain–temperature history. And while $\|E_e\|$ may be small compared to unity, $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ could be sufficiently large relative to $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ that the term $E_e\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) - \mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)E_e$ in (7.13) is of the same order as $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$, in which case the simpler approximation (7.12)₁ for $\tilde{D}_p$ would not be valid. In this case Classes II and III are not approximately equivalent for small elastic shear strains. This follows from (7.13) and the fact that $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ depends on V_p as well as the state variables for flow rules in Class II (see Section 6.2).

On the other hand, there would seem to be no physical basis for plastic spin unaccompanied by changes in plastic strain, so it is not unreasonable to expect that the plastic spin $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ is bounded by the plastic strain rate $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$, in the sense that

$$\left\|\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)\right\| \leqslant \alpha_1 \left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\| \tag{7.16}$$

for some constant α_1. If such a bound holds, then by (7.15) with $H = \mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$, we have

$$\left\|E_e\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) - \mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)E_e\right\| \leqslant 2\alpha_1 \|E_e\| \cdot \left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\|. \tag{7.17}$$

If the constant α_1 in (7.16) is sufficiently small, say on the order of unity, then the left-hand side of (7.17), and hence $\left\|E_e\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) - \mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)E_e\right\|$, will be small relative to $\left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\|$ whenever $\|E_e\|$ is small relative to unity, that is, for small elastic shear strains. In this case the simpler approximation (7.12)₁ for $\tilde{D}_p$ follows from (7.13).

Regarding the approximation (7.14) for $\tilde{W}_p$, we see that if $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ is sufficiently large relative to $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$, then the term $E_e\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) - \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)E_e$ could be as large or larger than $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$, in which case the simpler approximation (7.12)₂ for $\tilde{W}_p$ would not be valid. Note that the bound (7.16) does not rule out this possibility. On the other hand, suppose there is a constant β_1 such that

$$\left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\| \leqslant \beta_1 \left\|\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)\right\|. \tag{7.18}$$
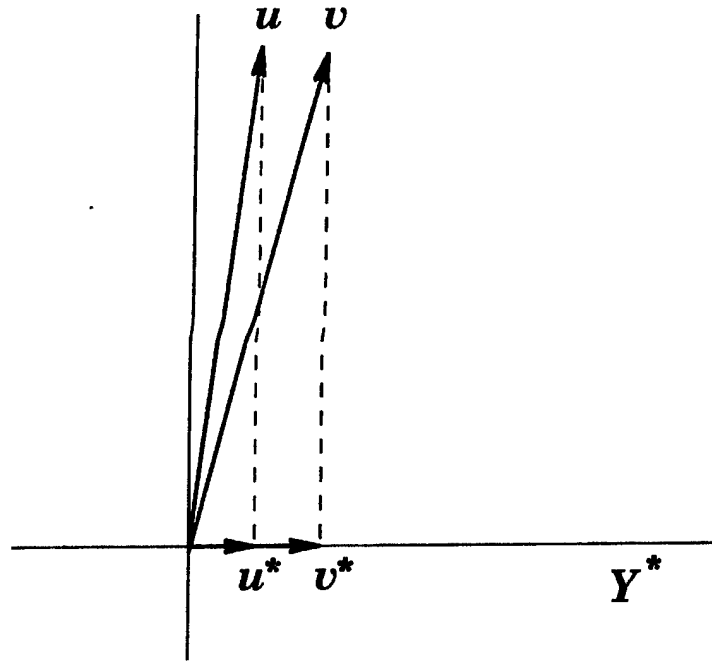
Fig. 1. Projection of two approximately equal vectors u and v onto a subspace Y^* of the vector space Y.

Arguing as above, we see that if β_1 is sufficiently small then the simpler approximation $(7.12)_2$ for $\tilde{W}_p$ would follow from (7.14).

7.4. Approximations for $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$

The approximations (7.13) and (7.14) for $\tilde{D}_p$ and $\tilde{W}_p$ in terms of $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ yield approximations for $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ in terms of $\tilde{D}_p$ and $\tilde{W}_p$:

$$\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) \approx \left(\frac{\rho_R}{\rho}\right)^{-2/3}\left(\tilde{D}_p - E_e\tilde{W}_p + \tilde{W}_p E_e\right), \tag{7.19}$$

and

$$\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) \approx \left(\frac{\rho_R}{\rho}\right)^{-2/3}\left(\tilde{W}_p - E_e\tilde{D}_p + \tilde{D}_p E_e\right). \tag{7.20}$$

To obtain (7.19), solve (7.13) for $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and (7.14) for $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$, and then substitute the second of these relations into the first. The result is (7.19) plus some terms of order $\left\|E_e\right\|^2\left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\|$, which are negligible relative to $\left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\|$ for small elastic shear strains. The derivation of (7.20) is similar.

If there is a constant $\tilde{\alpha}_1$ such that

$$\left\|\tilde{W}_p\right\| \leqslant \tilde{\alpha}_1\left\|\tilde{D}_p\right\| \tag{7.21}$$

then on using (7.19) and arguing as in Section 7.3, we see that for small elastic shear strains the simpler approximation $(7.12)_1$ for $\tilde{D}_p$ is valid provided the constant $\tilde{\alpha}_1$ is sufficiently small, say on the order of unity. Similarly, if there is a constant $\tilde{\beta}_1$ such that

$$\left\| \tilde{D}_p \right\| \leqslant \tilde{\beta}_1 \left\| \tilde{W}_p \right\|, \qquad (7.22)$$

then on using (7.20) we see that for small elastic shear strains the simpler approximation $(7.12)_2$ for $\tilde{W}_p$ is valid provided the constant $\tilde{\beta}_1$ is sufficiently small.

Although bounds on the plastic strain rate of the form (7.22) or (7.18) might hold for some deformations, they would seem to be too restrictive to impose as a general constitutive requirement. In particular, (7.18) implies that the plastic strain rate $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ is zero whenever the plastic spin $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ is zero, while (7.22) implies that the plastic strain rate $\tilde{D}_p$ is zero whenever the plastic spin $\tilde{W}_p$ is zero.

7.5. Equivalence of the bounds on the plastic spin

The bounds (7.16) and (7.21) on the plastic spin involve the tensors $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ and the tensors $\tilde{D}_p$ and $\tilde{W}_p$, respectively. Recall that these are pairs of preferred plastic strain rate and plastic spin tensors for the flow rules in Class I. It is of some interest to consider analogous bounds on the plastic spin involving the preferred measures for Classes II and III:

$$\left\| \Omega_p \right\| \leqslant \alpha_2 \left\| \mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) \right\|, \qquad (7.23)$$

and

$$\left\| \Omega_p \right\| \leqslant \alpha_3 \left(\frac{\rho_R}{\rho}\right)^{-2/3} \left\| \tilde{D}_p \right\|. \qquad (7.24)$$

We claim that for small elastic shear strains, the bounds (7.16), (7.21), (7.23), and (7.24) on the plastic spins are approximately equivalent if the coefficients α_1, $\tilde{\alpha}_1$, α_2, and α_3 are on the order of unity. From this result and the discussion at the end of Section 7.2, we may conclude the following. *If the material is such that one of the aforementioned bounds holds with the appropriate coefficient α on the order of unity, then for small elastic shear strains the Classes II and III are approximately equivalent.*

The remainder of this section is devoted to a proof of the above claim. First note that by the inequality (6.14), the bound (7.23) implies the bound (7.16) with $\alpha_1 = \alpha_2 + 1$. Conversely, by the inequality (6.15), the bound (7.16) implies (7.23) with

$\alpha_2 = \alpha_1 + 1$. And in either case α_1 is on the order of unity iff α_2 is. Note that these results hold for arbitrary elastic strains.

For the remainder of this discussion we assume small elastic shear strains. Then, as noted previously, the bound (7.16) with α_1 on the order of unity implies the approximation (7.12)$_1$ for $\tilde{D}_p$, though not necessarily the approximation (7.12)$_2$ for $\tilde{W}_p$. However, the approximation (7.14) for $\tilde{W}_p$ does hold, and the use of (7.16), (7.15), and (7.12)$_1$ in this approximation yields $\left\|\tilde{W}_p\right\| \leqslant \left(\alpha_1 + 2\left\|E_e\right\|\right)\left\|\tilde{D}_p\right\|$ (approximately), so that (7.21) holds with $\tilde{\alpha}_1 \approx \alpha_1$. Conversely, since the bound (7.21) with $\tilde{\alpha}_1$ on the order of unity also implies (7.12)$_1$, use of (7.21), (7.15), and (7.12)$_1$ in the approximate formula (7.20) for $\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)$ yields (7.16) with $\alpha \approx \tilde{\alpha}_1$. Thus we have established the equivalence of the bounds (7.16), (7.21), and (7.23). Next, suppose that the bound (7.23) holds with α_2 on the order of unity. Since this implies the bound (7.16) with α_1 on the order of unity, which in turn implies the approximation (7.12)$_1$ for $\tilde{D}_p$, use of (7.12)$_1$ in (7.23) yields (7.24) with $\alpha_3 \approx \alpha_2$.

Finally, suppose that the bound (7.24) holds with α_3 on the order of unity. To complete the proof of the claim, we show that this condition implies the bound (7.16) with $\alpha_1 \approx \alpha_3 + 1$. By the inequality (6.14), we see that (7.24) implies

$$\left\|\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right)\right\| \leqslant \alpha_3\left(\frac{\rho_R}{\rho}\right)^{-2/3}\left\|\tilde{D}_p\right\| + \left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\|.$$

Then the desired result follows provided that we can use the approximation (7.12)$_1$ for $\tilde{D}_p$. However, we have not yet shown that (7.12)$_1$ follows from (7.24) (with α_3 on the order of unity), so it remains to establish this result. From the approximation (7.13) for $\tilde{D}_p$ and the identity (6.1), we obtain

$$\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) + E_e \mathbb{M}_{V_p}\left[\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right] - \mathbb{M}_{V_p}\left[\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right] E_e$$

$$\approx \left(\frac{\rho_R}{\rho}\right)^{-2/3}\tilde{D}_p - E_e \Omega_p + \Omega_p E_e. \tag{*}$$

Since, by (A.7), $\left\|\mathbb{M}_{V_o}\left[\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right]\right\| \leqslant \left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\|$, we see that

$$\left\|E_e \mathbb{M}_{V_p}\left[\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right] - \mathbb{M}_{V_p}\left[\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right] E_e\right\| \leqslant 2\|E_e\| \cdot \left\|\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)\right\|.$$

And by (7.24),

$$\left\|-E_e \Omega_p + \Omega_p E_e\right\| \leqslant 2\|E_e\| \cdot \|\Omega_p\| \leqslant 2\alpha_3\left(\frac{\rho_R}{\rho}\right)^{-2/3}\left\|\tilde{D}_p\right\|.$$

Thus, for small elastic shear strains, the terms involving E_e on the left side of (*) are small relative to $\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right)$. If, in addition, α_3 is on the order of unity, then the

terms involving E_e on the right side of (*) are small relative to $(\rho_R/\rho)^{-2/3}\tilde{D}_p$. The approximation $(7.12)_1$ for $\tilde{D}_p$ follows, which completes the proof of the claim.

8. Inconsistent flow rules

In view of the polar decompositions $F_p = R_p U_p = V_p R_p$, some authors have chosen to describe the evolution of F_p by means of separate evolution equations for R_p and either U_p or V_p. In this section we discuss some inconsistencies in certain flow rules of this type. In the next section we consider a consistent class of flow rules described by evolution equations for V_p and R_p.

We begin by noting that flow rules of the form

$$\dot{U}_p U_p^{-1} = \mathcal{L}_p(\mathcal{S}), \qquad R_p \equiv I \tag{8.1}$$

are occasionally encountered in the literature. Since $R_p \equiv I$ implies $F_p \equiv U_p \equiv V_p$, the flow rule (8.1) can just as well be written as

$$\dot{V}_p V_p^{-1} = \mathcal{L}_p(\mathcal{S}), \qquad R_p \equiv I. \tag{8.2}$$

More generally, one could consider a class of flow rules of the form

$$\dot{V}_p V_p^{-1} = \mathcal{L}_p(\mathcal{S}), \qquad \Omega_p = \mathcal{W}_p(\mathcal{S}). \tag{8.3}$$

Recalling that $\Omega_p = \dot{R}_p R_p^T$ and that $R_p = I$ initially, we see that (8.3) reduces to (8.2) if the function $\mathcal{W}_p \equiv 0$. Since $\det V_p = \det U_p = 1$, the tensors $\dot{V}_p V_p^{-1}$ and $\dot{U}_p U_p^{-1}$ are deviatoric. Also note that $\dot{V}_p V_p^{-1}$ and $\dot{U}_p U_p^{-1}$ are generally not symmetric. Thus the range of the function $\mathcal{L}_p$ in (8.1)–(8.3) should be the set of all deviatoric, second-order tensors.

It turns out that the (equivalent) flow rules (8.1) and (8.2) are inconsistent with the assumption that F_p (and hence U_p and V_p) is not a state variable. The same conclusion holds for all flow rules of the form (8.3), regardless of the choice of the skew function $\mathcal{W}_p$. Indeed, the inconsistency is inherent in the relation

$$\dot{V}_p V_p^{-1} = \mathcal{L}_p^+(\mathcal{S}) \qquad \text{or} \qquad \dot{U}_p U_p^{-1} = \mathcal{L}_p(\mathcal{S}) \tag{8.4}$$

itself, and the following discussion is independent of any constitutive assumptions on the plastic spin.

First, consider the evolution equation $(8.4)_1$ for V_p. If we set $\mathcal{L}_p^+ = \text{sym}\mathcal{L}_p$ and $\mathcal{L}_p^- = \text{skw}\mathcal{L}_p$, then $(8.4)_1$ is equivalent to the conditions

$$\text{sym}\left(\dot{V}_p V_p^{-1}\right) = \mathcal{L}_p^+(\mathcal{S}), \qquad \text{skw}\left(\dot{V}_p V_p^{-1}\right) = \mathcal{L}_p^-(\mathcal{S}). \tag{8.5}$$

Since the tensor $\dot{V}_p V_p^{-1}$ is deviatoric, so is $\mathrm{sym}\left(\dot{V}_p V_p^{-1}\right)$. From the algebraic identity (6.3) with $B = \dot{V}_p$ and $A = V_p$, we see that the skew part of $\dot{V}_p V_p^{-1}$ is determined by V_p and the symmetric part of $\dot{V}_p V_p^{-1}$ through the compatibility condition

$$\mathrm{skw}\left(\dot{V}_p V_p^{-1}\right) = \mathbb{M}_{V_p}\left[\mathrm{sym}\left(\dot{V}_p V_p^{-1}\right)\right].\tag{8.6}$$

When (8.5) is substituted into the compatibility condition (8.6), we obtain

$$\mathcal{L}_p^-(\mathcal{S}) = \mathbb{M}_{V_p}\left[\mathcal{L}_p^+(\mathcal{S})\right].\tag{8.7}$$

The left-hand side of (8.7) depends only on the state variables $\mathcal{S}$, whereas the right-hand side also depends nontrivially on V_p.[18] This is a contradiction since V_p does not belong to the list $\mathcal{S}$ of state variables. *It follows that (8.4)$_1$ is inconsistent with the assumption that V_p is not a state variable.* This is a consequence of the fact that the skew part of $\dot{V}_p V_p^{-1}$ is determined by V_p and the symmetric part of $\dot{V}_p V_p^{-1}$ through the compatibility condition (8.6).

Similarly, the evolution equation (8.4)$_2$ is inconsistent with the assumption that U_p is not a state variable. This is a consequence of the fact that the skew part of $\dot{U}_p U_p^{-1}$ is determined by U_p and the symmetric part of $\dot{U}_p U_p^{-1}$ through the compatibility condition (6.7).

The inconsistency in the evolution equation (8.4)$_1$ for V_p could be eliminated by allowing $\mathcal{L}_p$ to depend on V_p, so that

$$\dot{V}_p V_p^{-1} = \mathcal{L}_p(\mathcal{S}, V_p).\tag{8.8}$$

However, in view of the compatibility condition (8.6), we are only free to postulate a constitutive relation for the symmetric part of $\dot{V}_p V_p^{-1}$. *Thus the function $\mathcal{L}_p$ in (8.8) cannot be prescribed arbitrarily but instead must satisfy the compatibility condition*

$$\mathrm{skw}\mathcal{L}_p(\mathcal{S}, V_p) = \mathbb{M}_{V_p}[\mathrm{sym}\mathcal{L}_p(\mathcal{S}, V_p)].\tag{8.9}$$

Similarly, the inconsistency in the evolution equation (8.4)$_2$ for U_p could be eliminated by allowing $\mathcal{L}_p$ to depend on U_p, so that

$$\dot{U}_p U_p^{-1} = \mathcal{L}_p(\mathcal{S}, U_p).\tag{8.10}$$

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[18] The right-hand side of (8.7) is independent of V_p iff $\mathcal{L}_p^+(\mathcal{S})$ is spherical, in which case $\mathbb{M}_{V_p}\left[\mathcal{L}_p^+(\mathcal{S})\right] = 0$; see Appendix A. Also, since $\mathcal{L}_p^+(\mathcal{S})$ is deviatoric, it is spherical iff it is 0. It follows that the right-hand side of (8.7) is independent of V_p iff $\mathcal{L}_p^+(\mathcal{S})$ and $\mathcal{L}_p^-(\mathcal{S})$ are both 0, i.e., if $\mathcal{L}_p(\mathcal{S}) = 0$, in which case (8.4)$_1$ implies that V_p is constant.

However, in view of the compatibility condition (6.7), we are only free to postulate a constitutive relation for the symmetric part of $\dot{U}_p U_p^{-1}$. *Thus the function $\mathcal{L}_p$ in* (8.10) *cannot be prescribed arbitrarily but instead must satisfy the compatibility condition*

$$\mathrm{skw}\mathcal{L}_p(\mathcal{S}, U_p) = \mathbb{M}_{U_p}[\mathrm{sym}\mathcal{L}_p(\mathcal{S}, \mathsf{U}_p)]. \tag{8.11}$$

This result has been obtained previously by Nemat-Nasser (1990, 1992)[19] and (for the special case where $R_p \equiv I$) by Obata et al. (1990).

The inconsistencies discussed above are not restricted to the choice of U_p or V_p as the tensor measure of plastic stretch. Let the symmetric positive-definite tensor G_p be a function of U_p or V_p (e.g., $G_p = U_p^2$ or $G_p = V_p^2$). Then analogous problems arise for flow rules that postulate a constitutive relation for $\dot{G}_p G_p^{-1}$. Indeed, by setting $B = \dot{G}_p$ and $A = G_p$ in the identity (6.3), we obtain the compatibility condition

$$\mathrm{skw}\left(\dot{G}_p G_p^{-1}\right) = \mathbb{M}_{G_p}\left[\mathrm{sym}\left(\dot{G}_p G_p^{-1}\right)\right], \tag{8.12}$$

which shows that $\mathrm{skw}\left(\dot{G}_p G_p^{-1}\right)$ is completely determined by G_p and $\mathrm{sym}\left(\dot{G}_p G_p^{-1}\right)$. Note that this issue of compatibility does not arise for the skew part of the plastic velocity gradient $\dot{F}_p F_p^{-1}$, since F_p is not necessarily symmetric.

We close this section with the following observation. For flow rules of the form (8.8) or (8.10), we may impose the additional constitutive assumption that $\mathrm{skw}\mathcal{L}_p \equiv 0$, equivalently, that $\dot{V}_p V_p^{-1}$ or $\dot{U}_p U_p^{-1}$ is symmetric for all motions, but in this case the restrictions imposed by the compatibility condition (8.9) or (8.11) are so severe that the flow rule is not physically reasonable. Indeed, consider (8.8) with $\mathrm{skw}\mathcal{L}_p \equiv 0$. Then the compatibility condition reduces to $0 = \mathbb{M}_{V_p}\left[\dot{V}_p V_p^{-1}\right]$. But by (A.9), this holds iff V_p commutes with $\dot{V}_p V_p^{-1}$, and hence with $\dot{V}_p$. This implies that the principal axes of V_p are necessarily fixed on any time interval for which the principal values of V_p are distinct. Since this restriction must hold regardless of any changes in the principal axes of E_e or $\tilde{T}$, the resulting flow rule is physically unreasonable.

9. Flow rules in Class IV

In this section we continue the approach of describing the evolution of F_p by separate evolution equations for V_p and R_p. By differentiating the polar decomposition $F_p = V_p R_p$, we obtain the kinematic relation

$$\dot{F}_p F_p^{-1} = \dot{V}_p V_p^{-1} + V_p \Omega_p V_p^{-1}. \tag{9.1}$$

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[19] However, he appears to conclude from his results that no constitutive relation needs to be specified for the plastic spin. This is not true in general, although as discussed in Section 4.6, it would be appropriate for isotropic materials with only scalar internal variables.

This suggests a class of flow rules of the form

$$\dot{V}_p V_p^{-1} = \mathcal{L}_p(\mathcal{S}, V_p), \qquad \Omega_p = \mathcal{W}_p(\mathcal{S}). \tag{9.2}$$

As noted in the previous section, the dependence of $\mathcal{L}_p$ on V_p cannot be omitted if, as assumed here, V_p is not a state variable. Furthermore, only the symmetric part of the function $\mathcal{L}_p$ can be prescribed arbitrarily; the skew part of $\mathcal{L}_p$ is determined by the compatibility condition (8.9).

The issue of compatibility of the skew part of $\mathcal{L}_p$ can be avoided if we replace $(9.2)_1$ by a constitutive relation for the symmetric part of $\dot{V}_p V_p^{-1}$ only. And since we are free to assume that $\mathrm{sym}\left(\dot{V}_p V_p^{-1}\right)$ is independent of V_p, we consider a class of flow rules of the form

$$\textbf{Class IV:} \qquad \mathrm{sym}\left(\dot{V}_p V_p^{-1}\right) = \mathcal{D}_p(\mathcal{S}), \qquad \Omega_p = \mathcal{W}_p(\mathcal{S}). \tag{9.3}$$

This class of flow rules is of the general form (3.4) with preferred measure of plastic strain rate $\mathbb{D}_p = \mathrm{sym}\left(\dot{V}_p V_p^{-1}\right)$ and preferred measure of plastic spin $\mathbb{W}_p = \Omega_p$. Thus the first criterion in Section 3 is satisfied. From $(9.3)_1$ and the compatibility condition (8.6), we see that the skew part of $\dot{V}_p V_p^{-1}$ is given by

$$\mathrm{skw}\left(\dot{V}_p V_p^{-1}\right) = \mathbb{M}_{V_p}[\mathcal{D}_p(\mathcal{S})]. \tag{9.4}$$

From (9.3) and (9.4) it follows that the flow rules in Class IV are of the form (9.2) with

$$\mathcal{L}_p(\mathcal{S}, V_p) = \mathcal{D}_p(\mathcal{S}) + \mathbb{M}_{V_p}[\mathcal{D}_p(\mathcal{S})]. \tag{9.5}$$

Of course, not every flow rule of the form (9.2) belongs to Class IV, since (9.2) allows for the possibility that $\mathrm{sym}\left(\dot{V}_p V_p^{-1}\right)$ depends on V_p as well as the state variables.

The main properties of flow rules in Class IV are:

1. Classes I and IV are mutually exclusive.
2. In general, Classes II and IV are neither mutually exclusive nor equivalent, that is, they have some but not all flow rules in common. Isotropic materials with only scalar internal variables are an exception, since Classes II and IV coincide for this special case.
3. Classes III and IV are mutually exclusive.
4. The flow rules in Class IV satisfy the third criterion in Section 3.
5. For those flow rules in Class IV which do not belong to Class II, the plastic stress power generally depends on the plastic stretch V_p as well as the state variables.

In view of this last result, Class IV would seem to be of limited interest. The remainder of this section is devoted to proving the above results.

First, recall that for flow rules in Class I, Ω_p depends on the plastic stretch V_p as well as the state variables (see Section 6). Since such a flow rule cannot belong to Class IV, the Classes I and IV are mutually exclusive. This is the first result above.

To establish the second result, note that the flow function $\mathcal{W}_p$ in (9.3) may be chosen to be identically zero, yielding a subclass of flow rules for which

$$\mathrm{sym}\left(\dot{V}_p V_p^{-1}\right) = \mathcal{D}_p(\mathcal{S}), \qquad \Omega_p \equiv 0. \tag{9.6}$$

Likewise, referring to the relations (4.6) characterizing flow rules in Class II, we may take the flow function $\mathcal{W}_p$ to be identically zero, yielding a subclass of flow rules for which

$$\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) = \mathcal{D}_p(\mathcal{S}), \qquad \Omega_p \equiv 0. \tag{9.7}$$

And since $\Omega_p \equiv 0$ implies $R_p \equiv I$ and hence $F_p \equiv U_p \equiv V_p$, it follows that the subclass of flow rules in Class IV for which $\Omega_p \equiv 0$ coincides with the subclass of flow rules in Class II for which $\Omega_p \equiv 0$. In particular, for isotropic materials with only scalar internal variables, Ω_p is necessarily zero for flow rules in Classes II–IV (see Section 4.6), so Classes II and IV coincide for this special case.

To show that there are flow rules in Class IV that do not belong to Class II, and also to establish the other results above, we need the following kinematic identities. The symmetric part of (9.1) yields the relation

$$\mathrm{sym}\left(\dot{F}_p F_p^{-1}\right) = \mathrm{sym}\left(\dot{V}_p V_p^{-1}\right) + \mathrm{sym}\left(V_p \Omega_p V_p^{-1}\right). \tag{9.8}$$

By taking the skew part of (9.1) and using the compatibility condition (8.6), we find that

$$\mathrm{skw}\left(\dot{F}_p F_p^{-1}\right) = \mathbb{M}_{V_p}\left[\mathrm{sym}\left(\dot{V}_p V_p^{-1}\right)\right] + \mathrm{skw}\left(V_p \Omega_p V_p^{-1}\right). \tag{9.9}$$

When these relations are substituted into the expressions (5.3)–(5.4) for $\tilde{D}_p$ and $\tilde{W}_p$, we obtain relations of the form

$$\tilde{D}_p = \tilde{D}_p^{\#}\left(\mathrm{sym}\left(\dot{V}_p V_p^{-1}\right), \Omega_p, C_e, V_p\right),$$

$$\tilde{W}_p = \tilde{W}_p^{\#}\left(\mathrm{sym}\left(\dot{V}_p V_p^{-1}\right), \Omega_p, C_e, V_p\right),$$

$$\tag{9.10}$$

where the dependence on V_p is nontrivial.

For a flow rule in Class IV, $\text{sym}\left(\dot{V}_p V_p^{-1}\right)$ and Ω_p depend only on the state variables, but by (9.8) we see that $\text{sym}\left(\dot{F}_p F_p^{-1}\right)$ will generally depend on V_p as well as the state variables.[20] Such a flow rule does not belong to Class II. It follows that Classes II and IV are neither equivalent nor mutually exclusive, that is, they have some but not all flow rules in common. As observed above, isotropic materials with only scalar internal variables are an exception to this statement.

Now consider the third and fourth results above. For a flow rule in Class IV, (9.9) implies that $\text{skw}\left(\dot{F}_p F_p^{-1}\right)$ depends on V_p as well as the state variables. And as noted above, $\text{sym}\left(\dot{F}_p F_p^{-1}\right)$ depends at most on V_p and the state variables. Consequently, $\dot{F}_p F_p^{-1}$ also depends only on V_p and the state variables. Thus, like Classes II and III, the flow rules in Class IV are of the general form (4.10). In particular, the third criterion in Section 3 is satisfied. From (9.10) it follows that $\tilde{D}_p$ and $\tilde{W}_p$ depend on V_p as well as the state variables for flow rules in Class IV. Since $\tilde{D}_p$ depends only on the state variables for flow rules in Class III, it follows that Classes III and IV are mutually exclusive. Thus the third and fourth results stated above are established.

To prove the fifth result, we must first determine which flow rules in Class IV do not belong to Class II. We have already shown that the flow rules in Class IV for which $\Omega_p \equiv 0$ belong to Class II; we claim that these are the only flow rules with this property. Indeed, from (4.6), (9.3), and (9.8), we see that a flow rule in Class IV belongs to Class II iff $\text{sym}\left(V_p \Omega_p V_p^{-1}\right)$ is a function of the state variables S only. And since Ω_p depends only on the state variables, this can occur iff V_p and Ω_p commute so that the V_p and V_p^{-1} terms cancel, in which case $\text{sym}\left(V_p \Omega_p V_p^{-1}\right) = \text{sym}(\Omega_p) = 0$. However, if Ω_p is not identically zero for all motions, then for flow rules in Classes II–IV we cannot require that V_p and Ω_p commute for all motions, since this would imply a dependence of Ω_p on V_p. Thus Classes II and IV coincide only on the subclass of flow rules with the property that Ω_p is identically zero for all motions.

Next, from (3.1) and (9.1) we see that the plastic stress power is given by

$$\rho_R \mathcal{P}_p^s = C_e \tilde{T} : \dot{V}_p V_p^{-1} + C_e \tilde{T} : V_p \Omega_p V_p^{-1}. \tag{9.11}$$

For flow rules in Class IV, the first term on the right depends only on the state variables, whereas the second term will generally depend on V_p as well, unless V_p and Ω_p commute. But as discussed in the previous paragraph, this latter condition can hold for all motions iff the flow rule is such that Ω_p is identically zero, in which case the flow rule belongs to Class II. Thus the plastic stress power generally

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[20] By (9.8) we also see that for flow rules in Class II, $\text{sym}\left(\dot{V}_p V_p^{-1}\right)$ generally depends on V_p as well as the state variables.

depends on the plastic stretch V_p for flow rules in Class IV that do not belong to Class II.[21] This conclusion holds even for small elastic shear strains and, in particular, even when the (approximately equivalent) bounds on the plastic spin discussed in Section 7.5 hold. In this case the approximation $(7.12)_1$ for $\tilde{D}_p$ in terms of $\text{sym}\left(\dot{F}_p F_p^{-1}\right)$ is valid, and the approximation

$$\rho_R \mathcal{P}_p^s \approx \left(\frac{\rho_R}{\rho}\right)^{2/3} \left[\tilde{T} : \text{sym}\left(\dot{V}_p V_p^{-1}\right) + \tilde{T} : \text{sym}\left(V_p \Omega_p V_p^{-1}\right)\right] \tag{9.12}$$

for the plastic stress power follows from (3.12), $(7.12)_1$, and (9.8). Arguing as above, we conclude that even for small elastic shear strains, the plastic stress power generally depends on the plastic stretch V_p for flow rules in Class IV that do not belong to Class II.

We close this section by noting the following kinematic relation for $\text{sym}\left(\dot{V}_p V_p^{-1}\right)$ in terms of $\text{sym}\left(\dot{F}_p F_p^{-1}\right)$, $\text{skw}\left(\dot{F}_p F_p^{-1}\right)$, and V_p:

$$\begin{aligned}
\text{sym}\left(\dot{V}_p V_p^{-1}\right) = &\,\text{sym}\left(\dot{F}_p F_p^{-1}\right) + \text{sym}\left\{V_p \mathbb{M}_{V_p}\left[\text{sym}\left(\dot{F}_p F_p^{-1}\right)\right]V_p^{-1}\right\} \\
&- \text{sym}\left\{V_p \text{skw}\left(\dot{F}_p F_p^{-1}\right)V_p^{-1}\right\}.
\end{aligned} \tag{9.13}$$

This is obtained by solving (6.1) for Ω_p and substituting the result into (9.8). For a flow rule in Class I, $\text{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\text{skw}\left(\dot{F}_p F_p^{-1}\right)$ depend only on the state variables, so by (9.13) it follows that $\text{sym}\left(\dot{V}_p V_p^{-1}\right)$ depends on V_p as well as the state variables for flow rules in Class I.

10. Summary and discussion

We have studied four representative classes of flow rules that satisfy the first and third criteria listed in Section 3. Each class is defined by requiring that some preferred measures for plastic strain rate and plastic spin depend only on the state variables. The plastic slip F_p is not regarded as a state variable. For the flow rules in each class the plastic velocity gradient $\dot{F}_p F_p^{-1}$ depends at most on the state variables and the plastic slip. In fact, for Classes II–IV it was shown that the dependence of $\dot{F}_p F_p^{-1}$ on F_p arises only through the left plastic stretch $V_p = \sqrt{F_p F_p^T}$, while for Class I the dependence of $\dot{F}_p F_p^{-1}$ on F_p is absent altogether, by assumption.

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[21] Note that even for those flow rules in Class IV which do belong to Class II, i.e., those flow rules satisfying (9.6), the plastic stress power may still depend on V_p unless the material is elastically isotropic (see Section 4.6).

We have shown that these four classes of flow rules are mutually exclusive, with the exception that Classes II and IV coincide on the subclass of flow rules for which the plastic spin Ω_p is identically zero. We also found that all flow rules in Classes I and III satisfy our second criterion, namely, that the plastic stress power depends only on the state variables. Flow rules in Classes II and IV generally violate this criterion because the plastic stress power generally depends on V_p as well as the state variables. However, for materials that are elastically isotropic, the criterion on the plastic stress power is indeed satisfied by all flow rules in Class II. On the other hand, even for elastically isotropic materials, the only flow rules in Class IV that satisfy the criterion on the plastic stress power are those for which Ω_p is identically zero. Since these flow rules also belong to Class II, Class IV would seem to be of limited interest.

For small elastic shear strains, we showed that Classes II and III are approximately equivalent provided that one of the four (approximately equivalent) bounds on the plastic spin discussed in Section 7.5 hold. In this case we may also conclude that

1. For all flow rules in Class II (with no restrictions on the material symmetry), the plastic stress power depends (approximately) only on the state variables,
2. For an isotropic material with only scalar internal variables, given any flow rule in one of the Classes I–III there are corresponding flow rules in the other two classes that produce, either exactly or approximately, the same Cauchy stress, other constitutive relations being the same.

The first result follows from the approximate equivalence of Classes II and III and the fact that flow rules in Class III satisfy the criterion on the plastic stress power without any restrictions. Regarding the second result, it was shown in Section 4.6 that for isotropic materials with only scalar internal variables, there are corresponding flow rules in Classes I and II that produce the same Cauchy stress, assuming all other constitutive relations are the same. Then the second result above follows from the approximate equivalence of Classes II and III.

The results in the preceding paragraph are generally not valid if only small elastic shear strains are assumed, that is, with no bounds on the plastic spin. Such bounds are not implied by any of the other criteria considered in this paper, nor do they follow from the plastic dissipation inequality (Scheidler and Wright, 2001), at least not without additional assumptions. On the other hand, there would seem to be no physical basis for plastic spin unaccompanied by changes in plastic strain, so the bounds on the plastic spin discussed in Section 7 do not seem unreasonable, even with the requirement that the coefficients α_i in these bounds be on the order of unity. Such bounds might be regarded as an additional criterion to be satisfied by all flow rules.

For reasons discussed above, of the four classes of flow rules introduced here only the first three would seem to be of interest. However, the criteria considered in this paper do not lead to a preference for one of these three classes over the others. The exception to this statement is that for materials with anisotropic elastic response, Class II may be less preferable since the plastic stress power depends on the plastic

stretch, although as discussed above this dependence is negligible if the elastic shear strains are sufficiently small and the plastic spin is bounded appropriately. In a follow-up paper we plan to examine other criteria that would be of assistance in selecting one of these classes over the others.

Appendix A

In this appendix we define the fourth-order tensors $\mathbb{L}_A$ and $\mathbb{M}_A$, discuss some of their properties, and give explicit formulas for $\mathbb{L}_A[H]$ and $\mathbb{M}_A[H]$.[22] Here A denotes a second-order, symmetric, positive-definite tensor, while H is an arbitrary second-order tensor unless specified otherwise.

The fourth-order tensor $\mathbb{L}_A$ is characterized by the property that the second-order tensor $X = \mathbb{L}_A[H]$ is the unique solution of the tensor equation

$$AX + XA = H. \tag{A.1}$$

In other words, $\mathbb{L}_A$ maps the tensor H on the right-hand side of Eq. (A.1) to the solution X of (A.1). Relative to a principal basis for A, the components of the solution X of (A.1) are $X_{ij} = H_{ij}/(a_i + a_j)$, where $a_i > 0$ are the principal values (eigenvalues) of A. Thus the ij component of $\mathbb{L}_A[H]$ is

$$(\mathbb{L}_A[H])_{ij} = \frac{H_{ij}}{a_i + a_j}, \tag{A.2}$$

relative to a principal basis for A. It follows that $\mathbb{L}_A$ maps symmetric tensors to symmetric tensors and skew tensors to skew tensors.

The fourth-order tensor $\mathbb{M}_A$ is characterized by the property that the second-order tensor $X = \mathbb{M}_A[H]$ is the unique solution of the tensor equation

$$AX + XA = AH - HA. \tag{A.3}$$

In other words, $\mathbb{M}_A$ maps the tensor H on the right-hand side of Eq. (A.3) to the solution X of (A.3). Since (A.3) is an equation of the form (A.1) with H replaced by $AH - HA$, it follows that

$$\mathbb{M}_A[H] = \mathbb{L}_A[AH - HA]. \tag{A.4}$$

Relative to a principal basis for A, the ij component of $\mathbb{M}_A[H]$ is

$$(\mathbb{M}_A[H])_{ij} = \frac{a_i - a_j}{a_i + a_j} H_{ij}. \tag{A.5}$$

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[22] See Scheidler (1994) for the derivation of results stated here without proof as well as for other component-free formulas.

If H is symmetric (skew), then $AH-HA$ is skew (symmetric), so by (A.4) and the properties of $\mathbb{L}_A$ noted above, or directly from (A.5), we see that $\mathbb{M}_A$ maps symmetric tensors to skew tensors and skew tensors to symmetric tensors. From (A.5) it follows that

$$\mathbb{M}_{\alpha A}[H] = \mathbb{M}_A[H], \qquad \forall \alpha > 0. \tag{A.6}$$

The norm of a tensor H, $\|H\|$, is defined by (6.12). Since the principal values a_i are positive, the coefficient of H_{ij} in (A.5) has absolute value less than one, and thus

$$\|\mathbb{M}_A[H]\| \leqslant \|H\| \tag{A.7}$$

for any tensor H and any symmetric positive-definite tensor A. Strict inequality holds in (A.7) whenever $H \neq 0$.

Since $X = 0$ is the unique solution of (A.1) when $H = 0$, it follows that

$$\mathbb{L}_A[H] = 0 \quad \text{iff} \quad H = 0. \tag{A.8}$$

When this result is applied to (A.4), we see that

$$\mathbb{M}_A[H] = 0 \quad \text{iff} \quad AH = HA. \tag{A.9}$$

For any orthogonal tensor Q, the tensor equation (A.1) is equivalent to

$$(QAQ^T)(QXQ^T) + (QXQ^T)(QAQ^T) = QHQ^T,$$

so it follows that $QXQ^T = \mathbb{L}_{QAQ^T}[QHQ^T]$. And since $X = \mathbb{L}_A[H]$, we see that

$$Q\mathbb{L}_A[H]Q^T = \mathbb{L}_{QAQ^T}[QHQ^T]. \tag{A.10}$$

Then (A.4) implies the analogous property (6.2) for $\mathbb{M}$. In other words, $\mathbb{L}_A[H]$ and $\mathbb{M}_A[H]$ are isotropic functions of A and H. It is easily shown that

$$\mathbb{L}_{A^{-1}}[H] = A\mathbb{L}_A[H]A, \qquad \mathbb{M}_{A^{-1}}[H] = -\mathbb{M}_A[H]. \tag{A.11}$$

The component formula (A.5) can be used to show that $\mathbb{M}_A[H]$ is in fact independent of A iff H is spherical, in which case $\mathbb{M}_A[H] = 0$. More precisely, $\mathbb{M}_A[H] = \mathbb{M}_B[H]$ for every symmetric positive definite A and B iff H is spherical. In view of (A.6), this conclusion remains valid if we add the restriction $\det A = \det B = 1$.

Let B be any symmetric tensor. To derive the relation

$$\text{skw}(BA^{-1}) = \mathbb{M}_A[\text{sym}(BA^{-1})] \tag{A.12}$$

used in the proof of the compatibility conditions (6.7) and (8.6), first verify the identity

$$A\mathrm{skw}\big(BA^{-1}\big) + \mathrm{skw}\big(BA^{-1}\big)A = A\mathrm{sym}\big(BA^{-1}\big) - \mathrm{sym}\big(BA^{-1}\big)A. \tag{A.13}$$

Then observe that this is an equation of the form (A.3) with $X = \mathrm{skw}(BA^{-1})$ and $H = \mathrm{sym}(BA^{-1})$, so the solution $X = \mathbb{M}_A[H]$ of (A.3) yields (A.12).

The simple component formulas (A.2) and (A.5) are only valid relative to a principal basis for A. Next, we list some of the simpler component-free formulas for $\mathbb{L}_A[H]$ and $\mathbb{M}_A[H]$.[23] Let I_A, II_A, and III_A denote the principal invariants of A, and define the symmetric tensor[24]

$$\tilde{A} = I_A I - A. \tag{A.14}$$

Then $\tilde{A}$ is coaxial with A and has principal values $\tilde{a}_i = a_j + a_k > 0$ (ijk distinct). Thus $\tilde{A}$ is also positive-definite with

$$\det\tilde{A} = I_A II_A - III_A = (a_1 + a_2)(a_2 + a_3)(a_3 + a_1) > 0. \tag{A.15}$$

When H is a skew tensor, $\mathbb{L}_A[H]$ is given by the simple formulas

$$\begin{aligned}
\big(\det\tilde{A}\big)\mathbb{L}_A[H] &= \tilde{A}H\tilde{A} \\
&= \big[(I_A)^2 - II_A\big]H - \big(A^2H + HA^2\big).
\end{aligned} \tag{A.16}$$

From (A.16)₁ and (A.4) it follows that for skew H, $\mathbb{M}_A[H]$ is given by the formulas

$$\begin{aligned}
\big(\det\tilde{A}\big)\mathbb{M}_A[H] &= \tilde{A}(AH - HA)\tilde{A} \\
&= A^2HA - AHA^2 + I_A\big(HA^2 - A^2H\big) + (I_A)^2(AH - HA).
\end{aligned} \tag{A.17}$$

In fact, it can be shown that (A.17) holds for any (not necessarily skew) tensor H.[25]

The formula (A.16) for $\mathbb{L}_A[H]$ does not hold for arbitrary H or even for symmetric H. However, it can be shown that for any tensor H,

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[23] Unlike the previous results, the component-free formulas listed here are valid only for tensors on a three-dimensional vector space.

[24] The notation $\tilde{A}$ for $I_A I - A$, which is consistent with the notation in Scheidler (1994), is not used outside this Appendix. In particular, the tilde on the tensors $\tilde{T}$, $\tilde{D}_p$, and $\tilde{W}_p$ in the main body of the paper is unrelated to the definition (A.14).

[25] The top formulas in (A.16) and (A.17) are due to Scheidler (1994). The bottom formula in (A.16) is due to Sidoroff (1978) and Guo (1984). The bottom formula in (A.17) is due to Mehrabadi and Nemat-Nasser (1987).

$$\mathbb{L}_A[H] = \frac{1}{2}A^{-1}(H + \mathbb{M}_A[H]) = \frac{1}{2}(H - \mathbb{M}_A[H])A^{-1}$$

$$= \frac{1}{4}(A^{-1}H + HA^{-1}) + \frac{1}{4}(A^{-1}\mathbb{M}_A[H] - \mathbb{M}_A[H]A^{-1}).$$

(A.18)

Since the formulas (A.17) for $\mathbb{M}_A[H]$ also hold for arbitrary H, they may be used in (A.18) to obtain component-free formulas for $\mathbb{L}_A[H]$ for arbitrary H.

Appendix B

In this Appendix we complete the derivation of the kinematic relations (5.5) and (5.6) for $\text{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\text{skw}\left(\dot{F}_p F_p^{-1}\right)$, give an alternate proof of the fundamental kinematic identity (6.1), and establish some properties of flow rules in Class III that were stated without proof in Section 4.2.

B1. Derivation of (5.5) and (5.6)

The formula (5.3) for $\tilde{D}_p$ is a tensor equation of the form (A.1) with

$$A = C_e, \qquad X = \text{sym}\left(\dot{F}_p F_p^{-1}\right),$$

$$H = 2\tilde{D}_p - \left\{C_e \text{skw}\left(\dot{F}_p F_p^{-1}\right) - \text{skw}\left(\dot{F}_p F_p^{-1}\right)C_e\right\}.$$

This equation has the solution

$$\text{sym}\left(\dot{F}_p F_p^{-1}\right) = \mathbb{L}_{C_e}[H]$$

$$= 2\mathbb{L}_{C_e}\left[\tilde{D}_p\right] - \mathbb{L}_{C_e}\left[C_e \text{skw}\left(\dot{F}_p F_p^{-1}\right) - \text{skw}\left(\dot{F}_p F_p^{-1}\right)C_e\right],$$

which, by (A.4) with $A = C_e$ and $H = \text{skw}\left(\dot{F}_p F_p^{-1}\right)$, yields the solution (5.5).

Similarly, the formula (5.4) for $\tilde{W}_p$ is a tensor equation of the form (A.1) with

$$A = C_e, \qquad X = \text{skw}\left(\dot{F}_p F_p^{-1}\right),$$

$$H = 2\tilde{W}_p - \left\{C_e \text{sym}\left(\dot{F}_p F_p^{-1}\right) - \text{sym}\left(\dot{F}_p F_p^{-1}\right)C_e\right\}.$$

This equation has the solution

$$\operatorname{sym}\left(\dot{F}_p F_p^{-1}\right) = \mathbb{L}_{C_e}[H]$$

$$= 2\mathbb{L}_{C_e}\left[\tilde{W}_p\right] - \mathbb{L}_{C_e}\left[C_e\operatorname{sym}\left(\dot{F}_p F_p^{-1}\right) - \operatorname{sym}\left(\dot{F}_p F_p^{-1}\right)C_e\right],$$

which, by (A.4) with $A = C_e$ and $H = \operatorname{sym}\left(\dot{F}_p F_p^{-1}\right)$, yields the solution (5.6).

B2. Alternate proof of (6.1)

On multiplying the kinematic relation (9.1) on the right by V_p, we obtain

$$\dot{V}_p = \left(\dot{F}_p F_p^{-1}\right)V_p - V_p\Omega_p = V_p\left(\dot{F}_p F_p^{-1}\right)^T + \Omega_p V_p, \qquad (B.1)$$

where the second expression follows by taking the transpose of the first and using the fact that $\dot{V}_p$ is symmetric. On decomposing $\dot{F}_p F_p^{-1}$ into its symmetric and skew parts and then rearranging (B.1)$_2$, we obtain the tensor equation

$$V_p\left\{\operatorname{skw}\left(\dot{F}_p F_p^{-1}\right) - \Omega_p\right\} + \left\{\operatorname{skw}\left(\dot{F}_p F_p^{-1}\right) - \Omega_p\right\}V_p$$

$$= V_p\operatorname{sym}\left(\dot{F}_p F_p^{-1}\right) - \operatorname{sym}\left(\dot{F}_p F_p^{-1}\right)V_p.$$

This is an equation of the form (A.3) with $A = V_p$, $X = \operatorname{skw}\left(\dot{F}_p F_p^{-1}\right) - \Omega_p$, and $H = \operatorname{sym}\left(\dot{F}_p F_p^{-1}\right)$. The solution $X = \mathbb{M}_A[H]$ of (A.3) yields (6.1).

B3. Some properties of flow rules in Class III

In Section 4.2 we stated that for flow rules in Class III, the tensors $\operatorname{sym}\left(\dot{F}_p F_p^{-1}\right)$, $\operatorname{skw}\left(\dot{F}_p F_p^{-1}\right)$, and $\tilde{W}_p$ depend on the plastic stretch V_p as well as the state variables. We prove these results here. The fact that $\operatorname{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\operatorname{skw}\left(\dot{F}_p F_p^{-1}\right)$ depend only on V_p and the state variables is the basis for the conclusion that all flow rules in Class III can be expressed in the form (4.10), i.e., that $\dot{F}_p F_p^{-1} = \hat{\Phi}_p(\mathcal{S}, V_p)$. As noted in Section 4.2, this implies that the flow rules in Class III satisfy the third criterion in Section 3.

On substituting the expression (9.1) for $\dot{F}_p F_p^{-1}$ into the definition (B.2)$_2$ of $\tilde{D}_p$, we find that

$$\operatorname{sym}\left(C_e \dot{V}_p V_p^{-1}\right) = \Sigma, \qquad \Sigma \equiv \tilde{D}_p - \operatorname{sym}\left(C_e \dot{V}_p \Omega_p V_p^{-1}\right). \qquad (B.2)$$

As shown below, the equation on the left may be solved for $\dot{V}_p V_p^{-1}$ in terms of Σ, U_e, and V_p:

$$\dot{V}_p V_p^{-1} = 2V_p U_e \mathbb{L}_{U_e V_p U_e} [U_e^{-1} \Sigma \, U_e^{-1}] U_e.$$ (B.3)

On taking the symmetric part of this relation and using the definition (B.2) of Σ , we obtain a relation of the general form

$$\text{sym}\left(\dot{V}_p V_p^{-1}\right) = \Upsilon_1(U_e, V_p)\left[\tilde{D}_p\right] - \Upsilon_2(U_e, V_p)[\Omega_p],$$ (B.4)

in which $\text{sym}\left(\dot{V}_p V_p^{-1}\right)$ depends linearly on $\tilde{D}_p$ and Ω_p and nonlinearly on U_e and V_p.

Now consider a flow rule in Class III. By assumption, $\tilde{D}_p$ and Ω_p depend only on the state variables. And since U_e is a state variable, it follows from (B.4) that $\text{sym}\left(\dot{V}_p V_p^{-1}\right)$ depends only on V_p and the state variables. This result, the relations (9.8) and (9.9), and the assumption that Ω_p depends only on the state variables imply that $\text{sym}\left(\dot{F}_p F_p^{-1}\right)$ and $\text{skw}\left(\dot{F}_p F_p^{-1}\right)$ also depend on V_p and the state variables for flow rules in Class III. Then by (5.4), we see that $\tilde{W}_p$ depends on V_p and the state variables.

To derive (B.3), first expand the left-hand side of (B.2)$_1$ to obtain the tensor equation

$$C_e\left(\dot{V}_p V_p^{-1}\right) + \left(\dot{V}_p V_p^{-1}\right)^T C_e = 2\Sigma.$$

This is not an equation of the general form (A.1) for the unknown $\dot{V}_p V_p^{-1}$ because of the presence of the transpose. We can get around this difficulty by multiplying both sides of this equation on the left and the right by $U_e^{-1}I$ and then rearranging to give

$$\left(U_e \dot{V}_p U_e\right)\left(U_e V_p U_e\right)^{-1} + \left(U_e V_p U_e\right)^{-1}\left(U_e \dot{V}_p U_e\right) = 2U_e^{-1}\Sigma \, U_e^{-1}.$$

This is an equation of the form (A.1) with $A = (U_e V_p U_e)^{-1}$, $X = U_e \dot{V}_p U_e$, and $H = 2U_e^{-1}\Sigma \, U_e^{-1}$. The solution is $X = \mathbb{L}_A[H]$, or

$$U_e \dot{V}_p U_e = 2\mathbb{L}_{(U_e V_p U_e)^{-1}}[U_e^{-1}\Sigma \, U_e^{-1}]$$
$$= 2(U_e V_p U_e)\mathbb{L}_{U_e V_p U_e}[U_e^{-1}\Sigma \, U_e^{-1}](U_e V_p U_e),$$

where the bottom relation follows from (A.11)$_1$ with $A = U_e V_p U_e$. After multiplying on the left by U_e^{-1} and on the right by $U_e^{-1} V_p^{-1}$, we obtain (B.3).

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